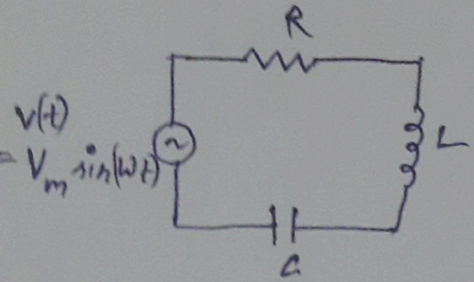
L-C-R circuit



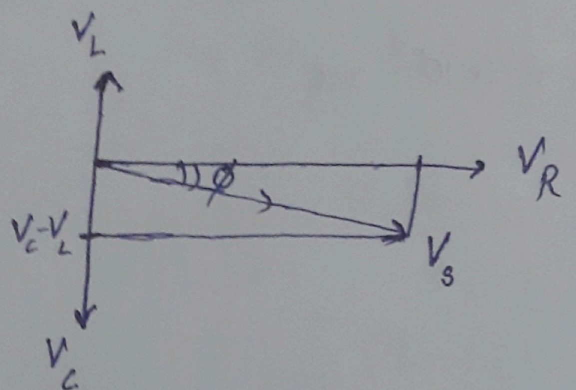
Impedance

$$Z = \sqrt{R^2 + (X_C - X_L)^2}$$

$X_C = \dfrac{1}{\omega C}$ ↳ Capacitive Reactance ; $X_L = \omega L$ ↳ Inductive Reactance

$I(t) = I_m \sin(\omega t + \phi)$ ↳ phase by which current leads the voltage

$I_m = \dfrac{V_m}{Z}$



$\phi \Rightarrow$ angle by which supply voltage lags the current
∴ Circuit is capacitive in nature.

Example 100 μF capacitor in series with a 40 Ω resistance, connected to a 110 V (rms)/60 Hz supply.
What is time lag between current max & voltage max?

60 Hz ⟶ $\omega = 2\pi \times 60 \approx 377$ rad/s

Capacitive reactance

$\dfrac{1}{\omega C} = \dfrac{1}{377 \times 10^{-4}} = 26.5\,\Omega$ $\qquad Z = \sqrt{40^2 + (26.5)^2} \approx 48\,\Omega$

$I_{rms} = \dfrac{110}{48} = 2.29\,A \Rightarrow I_m = 2.29\sqrt{2}$

$I_m = 3.24\,A$

Angle by which ~~cu~~ current leads voltage

$\phi = \tan^{-1}\dfrac{X_C}{R} = 0.56$ rad

Time lag b/w current & voltage max. (2)

$$\frac{\phi}{\omega} = 1.55ms$$

$f = 1\,kHz \rightarrow \omega = 6283$ rad/sec

$$\omega C = 6283 \times 10^{-4} = 0.63\,\Omega$$

$$X_c = \frac{1}{\omega C} = 1.59\,\Omega$$

$$Z = \sqrt{40^2 + (1.59)^2} = 40.03\,\Omega$$

$$\frac{X_c}{R} = \frac{1.59}{40} = 0.089 \Rightarrow \phi = \tan^{-1}\left(\frac{X_c}{R}\right) \approx 0.039\text{ rad}$$

Time lag $\frac{0.039}{6283} = 6.3 \times 10^{-6}$ sec

Analytical solution:

$$L\frac{dI}{dt} + IR + \frac{Q}{C} = V_m \sin \omega t$$

$$I = \frac{dQ}{dt}$$

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}.I = V_m \omega \cos(\omega t)$$

$$I = I_m \sin(\omega t + \phi)$$

$$\frac{dI}{dt} = I_m \omega \cos(\omega t + \phi)$$

$$\frac{d^2I}{dt^2} = -I_m \omega^2 \sin(\omega t + \phi)$$

$$I_m \omega \left[\underbrace{\left(-L\omega + \frac{1}{\omega C}\right)}_{X_C - X_L} \sin(\omega t + \phi) + R\cos(\omega t + \phi)\right] = V_m \omega \cos \omega t$$

$\cancel{I_m \omega [}$

$$R = A\cos\theta$$

$$X_C - X_L = A\sin\theta$$

$$A^2 = R^2 + (X_C - X_L)^2 = Z^2 \Rightarrow A = Z$$

$$\tan\theta = \frac{X_C - X_L}{R}$$

$$I_m Z \cos(\omega t + \phi - \theta) = V_m \cos(\omega t);$$

$$I_m Z = V_m \Rightarrow I_m = \frac{V_m}{Z}$$

$$\theta = \phi$$

$$I = \frac{V_m}{Z}\sin(\omega t + \phi) \; ; \; \tan\phi = \frac{X_C - X_L}{R}$$

Resonance

$$I_m = \frac{V_m}{\sqrt{R^2 + (X_C - X_L)^2}}$$

$$X_C = \frac{1}{\omega C} \quad ; \quad X_L = \omega L$$

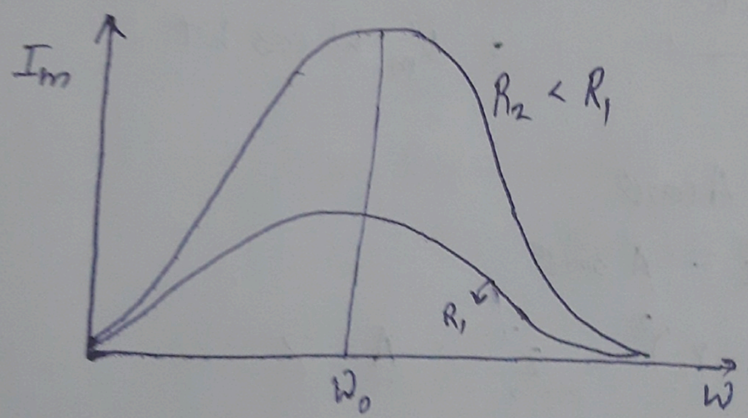
Z is minimum when $\frac{1}{\omega C} = \omega L$

$$\Rightarrow \omega = \omega_0 = \sqrt{\frac{1}{LC}}$$

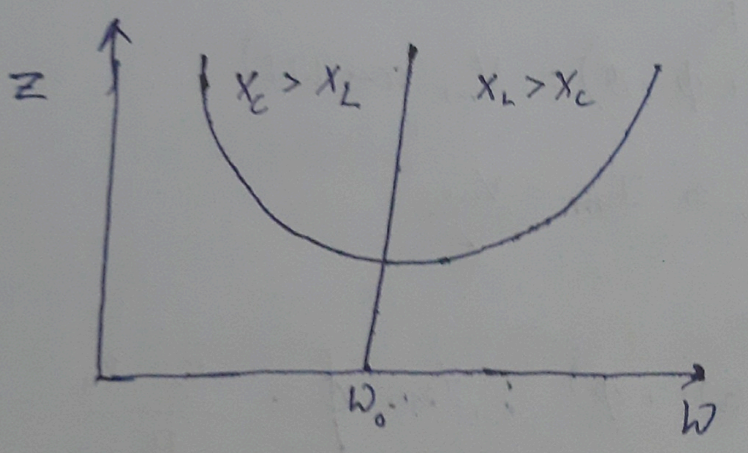
$\omega_0 \rightarrow$ Resonant frequency

At resonance

$$I_m = \frac{V_m}{R} \quad ; \text{ phase } \phi = 0$$



Both L & C must be present

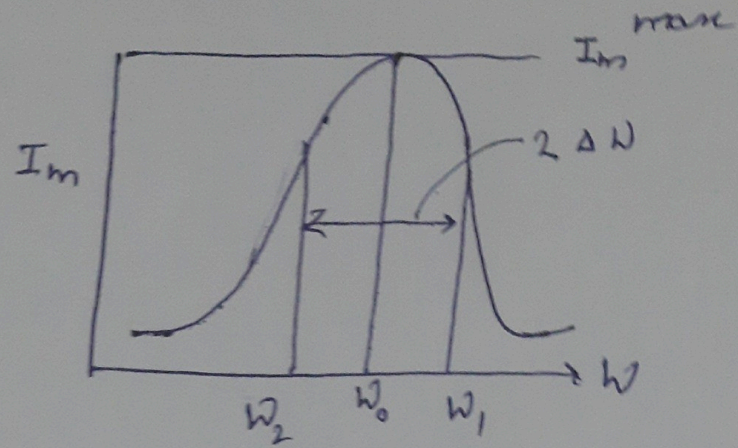


If $\omega^2 > \omega_0^2$

$$\omega^2 > \frac{1}{LC}$$

$$I \propto \frac{1}{Z}$$

$I^2 Z$ is maximum at $\omega = \omega_0$



Increase or decrease frequency (starting with W_0) till power absorbed is half the maximum (at W_0)

$2\Delta W = (W_1 - W_2)$

Band width (BW)

$P = 0.5 I^2 R$ → at half power pt.

$= \left(\frac{I}{\sqrt{2}}\right)^2 R$

Let

$W_1 = W_0 + \Delta W$

$W_2 = W_0 - \Delta W$

$2\Delta W =$ Band width

sharper resonance ⟶ Lower bandwidth

At $W = W_1 \Rightarrow I_m = \frac{V_m}{\sqrt{R^2 + \left(W_1 L - \frac{1}{W_1 C}\right)^2}} \equiv \frac{I_m^{max}}{\sqrt{2}}$

$= \frac{V_m}{R \cdot \sqrt{2}}$

By solving

$\Delta W = \frac{R}{2L}$

sharpness : Q → quality factor

$\frac{W_0}{2\Delta W} = \frac{W_0 L}{R} \equiv Q$