Q2) If $x = \operatorname{cosec}\theta - \sin\theta$ and $y = \operatorname{cosec}^n\theta - \sin^n\theta$, then show that

$$\left[(x^2+4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2+4)\right]$$

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Solution: As $x = \operatorname{cosec}\theta - \sin\theta$

$$\Rightarrow x^2+4 = (\operatorname{cosec}\theta - \sin\theta)^2 + 4 = (\operatorname{cosec}\theta + \sin\theta)^2 \quad \text{--- (1)}$$

$$y^2+4 = (\operatorname{cosec}^n\theta - \sin^n\theta)^2 + 4 = (\operatorname{cosec}^n\theta + \sin^n\theta)^2 \quad \text{--- (2)}$$

Now,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{n(\operatorname{cosec}^{n-1}\theta)(-\operatorname{cosec}\cot\theta) - n\sin^{n-1}\theta\cos\theta}{-\operatorname{cosec}\theta\cot\theta - \cos\theta}$$

$$= \frac{n(\operatorname{cosec}^n\theta\cot\theta + \sin^{n-1}\theta\cos\theta)}{(\operatorname{cosec}\theta\cot\theta + \cos\theta)}$$

$$= \frac{n(\operatorname{cosec}^n\theta + \sin^n\theta)}{(\operatorname{cosec}\theta + \sin\theta)} = \frac{n\sqrt{y^2+4}}{\sqrt{x^2+4}} \quad (\text{from } 1 \,\&\, 2)$$

Squaring Both sides we get,

$$\left[(x^2+4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2+4)\right] \quad \underline{\underline{\text{Ans}}}$$