4. Let $E_1$ and $E_2$ be two ellipses whose centers are at the origin. The major axes of $E_1$ and $E_2$ lie along the x-axis and the $y$-axis, respectively. Let S be the circle $x^2 + (y-1)^2 = 2$. The straight line $x + y = 3$ touches the curves $S$, $E_1$ and $E_2$ at $P$, $Q$ and R respectively. Suppose that $PQ = PR = \frac{2\sqrt{2}}{3}$. If $e_1$ and $e_2$ are the eccentricities of $E_1$ and $E_2$, respectively, then the correct expression(s) is (are) ***(JEE Adv. 2015)***

(a) $e_1^2 + e_2^2 = \frac{43}{40}$

(b) $e_1 e_2 = \frac{\sqrt{7}}{2\sqrt{10}}$

(c) $\left|e_1^2 - e_2^2\right| = \frac{5}{8}$

(d) $e_1 e_2 = \frac{\sqrt{3}}{4}$

**Solution: -**

4. (a, b)

Let $E_1 : \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where $a>b$

and $E_2 : \frac{x^2}{c^2}+\frac{y^2}{d^2}=1$ where $c<d$

Also $S : x^2+(y-1)^2=2$

Tangent at $P(x_1, y_1)$ to S is $x+y=3$

To find point of contact put $x=3-y$ in S. We get P(1, 2)

Writing eq$^n$ of tangent in parametric form

$$\frac{x-1}{\frac{-1}{\sqrt{2}}}=\frac{y-2}{\frac{1}{\sqrt{2}}}=\pm\frac{2\sqrt{2}}{3}$$

$$x=\frac{-2}{3}+1 \text{ or } \frac{2}{3}+1 \text{ and } y=\frac{2}{3}+2 \text{ or } \frac{-2}{3}+2$$

$$\Rightarrow \quad x=\frac{1}{3} \text{ or } \frac{5}{3} \text{ and } y=\frac{8}{3} \text{ or } \frac{4}{3}$$

$$\therefore \quad Q\left(\frac{5}{3},\frac{4}{3}\right) \text{ and } R\left(\frac{1}{3},\frac{8}{3}\right)$$

eq$^n$ of tangent to $E_1$ at Q is

$\frac{5x}{3a^2}+\frac{4y}{3b^2}=1$ which is identical to $\frac{x}{3}+\frac{y}{3}=1$

$$\Rightarrow \quad a^2=5 \text{ and } b^2=4 \Rightarrow e_1^2=1-\frac{4}{5}=\frac{1}{5}$$

eq$^n$ of tangent to $E_2$ at $R$ is

$\frac{x}{3c^2}+\frac{8y}{3d^2}=1$ identical to $\frac{x}{3}+\frac{y}{3}=1$

$$\Rightarrow \quad c^2=1,\ d^2=8 \Rightarrow e_2^2=1-\frac{1}{8}=\frac{7}{8}$$

$$\therefore \quad e_1^2+e_2^2=\frac{43}{40},\ e_1e_2=\frac{\sqrt{7}}{2\sqrt{10}},\ \left|e_1^2-e_2^2\right|=\frac{27}{40}$$