

# Differentiation Of Functions In Parametric Form

Sometimes,  $x$  &  $y$  are given as functions of single variables, i.e.,  $x = \phi(t)$ ,  $y = \psi(t)$  are two functions and  $t$  is a variable. In such a case  $x$  &  $y$  are called parametric functions or parametric equations and  $t$  is called the parameter.

To find  $\frac{dy}{dx}$  in case of parametric functions, we first obtain the relationship between  $x$  &  $y$  by eliminating the parameter  $t$ , and then we differentiate it with respect to  $x$ . But every time it is not convenient to eliminate the parameter. Therefore,  $\frac{dy}{dx}$  can also be obtained by the following formula.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Example 1  $\rightarrow$  Find  $\frac{dy}{dx}$  if  $x = a(\theta - \sin\theta)$  and  $y = a(1 - \cos\theta)$

Solution: We have  $x = a(\theta - \sin\theta)$  and  $y = a(1 - \cos\theta)$

$$\frac{dx}{d\theta} = a(1 - \cos\theta) \quad \frac{dy}{d\theta} = a \sin\theta$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{a \sin\theta}{a(1 - \cos\theta)} = \frac{2 \sin\theta/2 \cos\theta/2}{2 \sin^2\theta/2} = \cot\theta/2$$

Example 2  $\rightarrow$  If  $x = a \sec^3\theta$  &  $y = a \tan^3\theta$ , find  $\frac{dy}{dx} \Big|_{\theta = \pi/3}$

Solution: We have  $x = a \sec^3\theta$  and  $y = a \tan^3\theta$   
 $\frac{dx}{d\theta} = 3a \sec^2\theta \frac{d}{d\theta}(\sec\theta) = 3a \sec^2\theta \tan\theta$

$$\text{and } \frac{dy}{d\theta} = 3a \tan^2\theta \frac{d}{d\theta}(\tan\theta) = 3a \tan^2\theta \sec^2\theta$$

$$\therefore \frac{dy}{dx} = \frac{3a \tan^2\theta \sec^2\theta}{3a \sec^2\theta \tan\theta} = \tan\theta = \sin\theta$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a \tan^2 \theta \sec^2 \theta}{3a \sec^2 \theta \tan \theta} = \frac{\tan \theta}{\sec \theta} = \sin \theta$$

$$\therefore \left. \frac{dy}{dx} \right|_{\theta = \pi/3} = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

Example 3 > Let  $y = x^3 - 8x + 7$ ,  $x = f(t)$ . If  $\frac{dy}{dt} = 2$  and  $x = 3$  at  $t = 0$ , then find the value of  $\frac{dx}{dt}$  at  $t = 0$ .

Solution > We have  $y = x^3 - 8x + 7$   
It is given that when  $t = 0$ ,  $x = 3$ .  
 $\therefore$  when  $t = 0$ ,  $\frac{dy}{dx} = 3 \cdot 3^2 - 8 = 19$

$$\text{Also, } \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \text{--- (1)}$$

Since, when  $t = 0$ ,  $\frac{dy}{dx} = 19$  and  $\frac{dy}{dt} = 2$ ,

$\therefore$  from (1):

$$19 = \frac{2}{dx/dt} \Rightarrow \boxed{\frac{dx}{dt} = \frac{2}{19}}$$

## Differentiation Using Logarithm

If  $y = [f_1(x)] f_2(x)$  or  $y = f_1(x) f_2(x) f_3(x) \dots$

or  $y = \frac{f_1(x) f_2(x) f_3(x) \dots}{g_1(x) g_2(x) g_3(x) \dots}$ , then it is convenient to take the logarithm of the function first and then differentiate.

Example 1: If  $x^m y^n = (x+y)^{m+n}$ , prove that  $\frac{dy}{dx} = \frac{y}{x}$ .

Solution: We have  $x^m y^n = (x+y)^{m+n}$   $\dots \dots \dots \log = (m+n) \log(x+y)$

Now, taking log on both sides, we get  $\Rightarrow m \log x + n \log y = \dots$   
 Differentiating both sides w.r.t  $x$ , we get

$$m \frac{1}{x} + n \frac{1}{y} \frac{dy}{dx} = \frac{m+n}{x+y} \frac{d}{dx}(x+y)$$

$$\Rightarrow \left( \frac{m}{x} + \frac{n}{y} \right) \frac{dy}{dx} = \left( \frac{m+n}{x+y} \right) \left( 1 + \frac{dy}{dx} \right)$$

$$\left\{ \frac{n}{y} - \left( \frac{m+n}{x+y} \right) \right\} \frac{dy}{dx} = \frac{m+n}{x+y} - \frac{m}{x}$$

$$\left\{ \frac{nx + ny - my - ny}{y(x+y)} \right\} \frac{dy}{dx} = \left\{ \frac{mrx + nx - mx - my}{(x+y)x} \right\}$$

$$\frac{nx - my}{y(x+y)} \frac{dy}{dx} = \frac{nx - my}{(x+y)x} \Rightarrow \boxed{\frac{dy}{dx} = \frac{y}{x}}$$

Example 2: Find  $\frac{dy}{dx}$  for  $y = (\sin x)^{\log x}$

Solution: Let  $y = (\sin x)^{\log x}$

Then  $y = e^{\log x \log \sin x}$

[Refer to the hints for this approach, but first try on your own].

Differentiate both sides w.r.t  $x$ , we get,

$$\frac{dy}{dx} = e^{\log x \log \sin x} \frac{d}{dx} \{ \log x \log \sin x \}$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\log x} \times \left\{ \log \sin x \frac{d}{dx} (\log x) + \log x \frac{d}{dx} (\log \sin x) \right\}$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\log x} \left\{ \frac{\log \sin x}{x} + \cot x \log x \right\}$$

Example \*: If  $y = x^{x^{x^{\dots}}}$ ,  $\frac{dy}{dx} = ?$

Solution: Since by deleting a single term from an infinite series, it remains  $y = x^y$

Solution: Since by deleting a single term from ... same.

Therefore, the given function may be written as:

$$y = x^x$$

$$\Rightarrow \log y = y \log x \quad [\text{taking log both sides}]$$

$$\text{Differentiating } \Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{dy}{dx} \log x + \frac{y}{x}$$

$$\Rightarrow \left\{ \frac{dy}{dx} \frac{[1 - y \log x]}{y} = \frac{y}{x} \right\}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2}{x(1 - y \log x)}$$

\* Use OF logarithm to find sum of special series:

- • If  $x < 1$ , prove that

$$\left[ \frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \dots + \infty = \frac{1}{1-x} \right]$$

Solution: The given series is in the form:  $\frac{f_1'(x)}{f_1(x)} + \frac{f_2'(x)}{f_2(x)} + \dots + \infty$

- Then consider the product  $f_1(x) \times f_2(x) \times f_3(x) \dots f_n(x)$

$$\text{Now } (1-x)(1+x)(1+x^2)(1+x^4) \dots (1+x^{2^{n-1}})$$

$$= (1-x^2)(1+x^2) \dots (1+x^{2^{n-1}})$$

$$= (1-x^4) \dots (1+x^{2^{n-1}})$$

$$\vdots$$

$$= (1-x^{2^n})(1+x^{2^{n-1}}) = 1-x^{2^n}$$

Now when  $n \rightarrow \infty$   $x^{2^n} \rightarrow 0$  ( $\because x < 1$ )

$\therefore$  taking  $n \rightarrow \infty$  in (1), we get,

$$(1-x)(1+x)(1+x^2)(1+x^4)\dots = 1$$

Taking logarithm, we get

$$\log(1-x) + \log(1+x) + \log(1+x^2) + \log(1+x^4) + \dots = 0$$

Differentiate w.r.t.  $x$ , we get

$$-\frac{1}{1-x} + \frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \dots = 0$$

$$\Rightarrow \left[ \frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \dots = \frac{1}{1-x} \right] \underline{\underline{\text{Ans}}}$$