INVERSE TRIGONOMETRIC FUNCTIONS

* INVERTIBLE FUNCTION

A function $f: A \to B$ is invertible iff it is a bijection.
The inverse of $f = f^{-1}$. is defined as:

$$f^{-1}(y) = x \Rightarrow y = f(x).$$

Domain of f^{-1} = range of f
Range of f = Domain of f.

* INVERSE OF TRIGONOMETRIC FUNCTIONS

In case of trig. functions, the function isn't invertible in all their domains. So, we have to restrict their domain in such a way that they becomes bijections.

eg. $\sin x$. Domain $= (-\infty, \infty)$ Range $= [-1, 1]$
But it is only invertible in $[-\pi/2, \pi/2]$. So,
$\sin^{-1} x$ Domain $= [-1, 1]$ Range $= [-\pi/2, \pi/2]$.

$\sin x$ is also invertible in other domains like $[3\pi/2, 5\pi/2]$, $[-5\pi/2, -3\pi/2]$, etc.

	Function	Domain	Range.
1.	$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
2.	$\cos^{-1} x$.	$[-1, 1]$	$[0, \pi]$
3.	$\tan^{-1} x$	$(-\infty, \infty)$	$(-\pi/2, \pi/2)$
4.	$\cot^{-1} x$	$(-\infty, \infty)$	$(0, \pi)$
5.	$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$
6.	$\operatorname{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[-\pi/2, 0) \cup (0, \pi/2]$

$\sin^{-1} x$ has infinitiely many values for $[-1, 1]$ but the value which lie in the above written range are called principal values.
Similar is the case for other functions also.

meaning of Inverse: $\sin^{-1}x = \theta \Rightarrow x = \sin\theta$
where $-1 \le x \le 1$ and $-\pi/2 \le \theta \le \pi/2$.

Similarly in restricted domains & range.

$\cos^{-1}x = \theta \Rightarrow \cos\theta = x$

$\tan^{-1}x = \theta \Rightarrow \tan\theta = x$

$\cot^{-1}x = \theta \Rightarrow \cot\theta = x$

$\sec^{-1}x = \theta \Rightarrow \sec\theta = x$

$\operatorname{cosec}^{-1}x = \theta \Rightarrow \operatorname{cosec}\theta = x$.

* properties of Inverse Trig. functions.

1) In the restricted domains and range,

$\sin^{-1}(\sin\theta) = \theta$, $\cos^{-1}(\cos\theta) = \theta$, $\tan^{-1}(\tan\theta) = \theta$

$\sec^{-1}(\sec\theta) = \theta$, $\operatorname{cosec}^{-1}(\operatorname{cosec}\theta) = \theta$, $\cot^{-1}(\cot\theta) = \theta$

2) In the restricted domains and range,

$\sin(\sin^{-1}\theta) = \theta$, $\cos(\cos^{-1}\theta) = \theta$, $\tan(\tan^{-1}\theta) = \theta$

$\sec(\sec^{-1}\theta) = \theta$, $\operatorname{cosec}(\operatorname{cosec}^{-1}\theta) = \theta$, $\cot(\cot^{-1}\theta) = \theta$.

3) $\sin^{-1}x = \operatorname{cosec}^{-1}(1/x)$ $\quad$ $\operatorname{cosec}^{-1}x = \sin^{-1}(1/x)$

$\cos^{-1}x = \sec^{-1}(1/x)$ $\quad$ $\sec^{-1}x = \cos^{-1}(1/x)$

$\tan^{-1}x = \cot^{-1}(1/x)$ $\quad$ $\cot^{-1}x = \tan^{-1}(1/x)$

4) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}} = \cot^{-1}\dfrac{\sqrt{1-x^2}}{x} = \sec^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right)$

$= \operatorname{cosec}^{-1}\left(\dfrac{1}{x}\right)$.

$\cos^{-1}x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\dfrac{\sqrt{1-x^2}}{x} = \cot^{-1}\dfrac{x}{\sqrt{1-x^2}} = \sec^{-1}\left(\dfrac{1}{x}\right) = \operatorname{cosec}^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right)$

$\tan^{-1}x = \sin^{-1}\dfrac{x}{\sqrt{1+x^2}} = \cos^{-1}\dfrac{1}{\sqrt{1+x^2}} = \sec^{-1}\sqrt{1+x^2} = \operatorname{cosec}^{-1}\dfrac{\sqrt{1+x^2}}{x} = \cot^{-1}\left(\dfrac{1}{x}\right)$

5) $\sin^{-1}x + \cos^{-1}x = \pi/2$, $\quad -1 \le x \le 1$.

$\tan^{-1}x + \cot^{-1}x = \pi/2$, $\quad -\infty < x < \infty$

Date
Page

$\sec^{-1}x + \operatorname{cosec}^{-1}x = \pi/2$, $x \leq -1$ or $x \geq 1$.

6) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ if $xy < 1$

$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right) + \pi$ if $xy > 1$.

7) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

8) $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right)$ if $x, y \geq 0$, $x^2+y^2 \leq 1$.

$\sin^{-1}x \pm \sin^{-1}y = \pi - \sin^{-1}\left(x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right)$

if $x, y \geq 0$, $x^2+y^2 > 1$.

9) $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\left(xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right)$ if $x, y \geq 0$, $x^2+y^2 \leq 1$.

$\cos^{-1}x \pm \cos^{-1}y = \pi - \cos^{-1}\left(xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right)$ if $x, y \geq 0$, $x^2+y^2 > 1$.

10) $\sin^{-1}(-x) = -\sin^{-1}(x)$

$\cos^{-1}(-x) = \pi - \cos^{-1}(x)$

$\tan^{-1}(-x) = -\tan^{-1}x$

$\cot^{-1}(-x) = \pi - \cot^{-1}(x)$

11) $2\sin^{-1}x = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$2\cos^{-1}x = \cos^{-1}(2x^2-1)$

$2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

12) $3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$

$3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$

$3\tan^{-1}x = \tan^{-1}\left(\frac{3x - x^3}{1-3x^2}\right)$

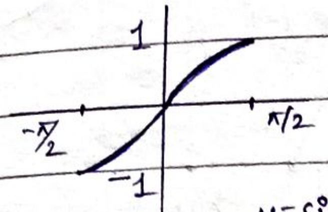
13) $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$

14) $\tan^{-1}x_1 + \tan^{-1}x_2 + \cdots = \tan^{-1}\left(\frac{S_1 - S_3 + S_5 - S_7 + \cdots}{1 - S_2 + S_4 - S_6 + \cdots}\right)$

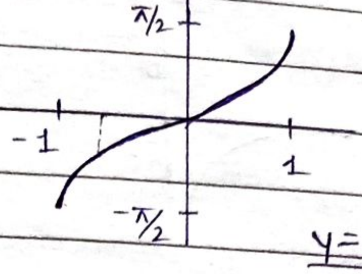
* Graphs of Inverse functions.

Inverses are images w.r.t $y = x$.

1. $\sin x$

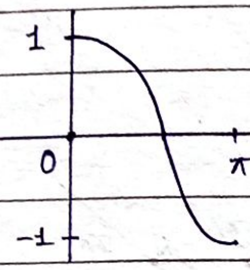


$y = \sin x$

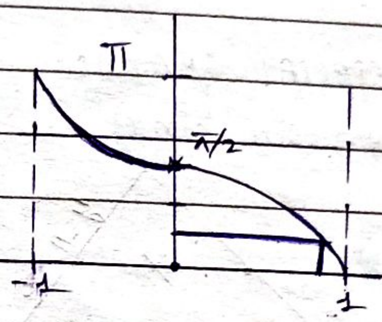


$y = \sin^{-1} x$.

2. $\cos x$

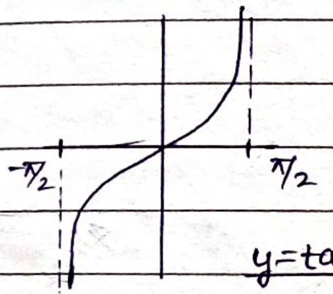


$y = \cos x$.

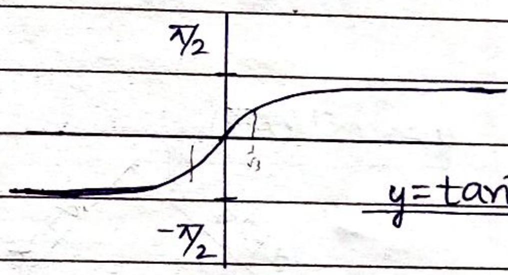


$y = \cos^{-1} x$

3. $\tan x$

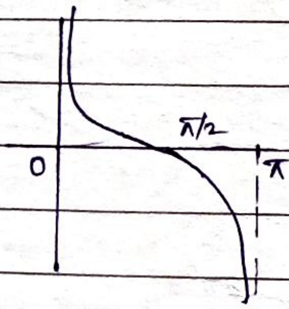


$y = \tan x$.

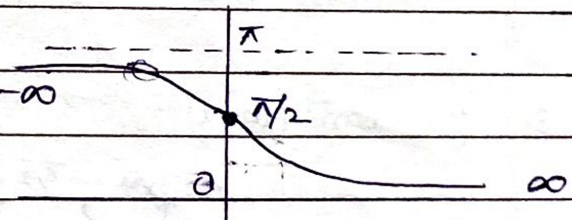


$y = \tan^{-1} x$.

4. $\cot x$

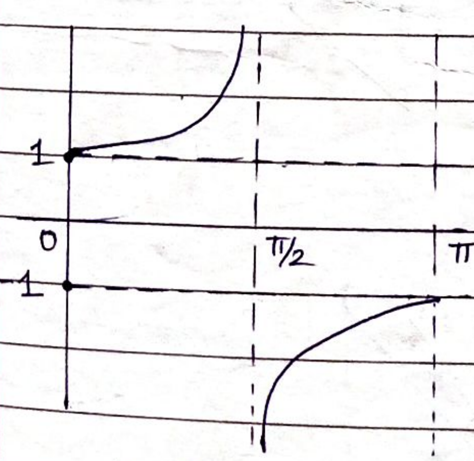


$y = \cot x$.

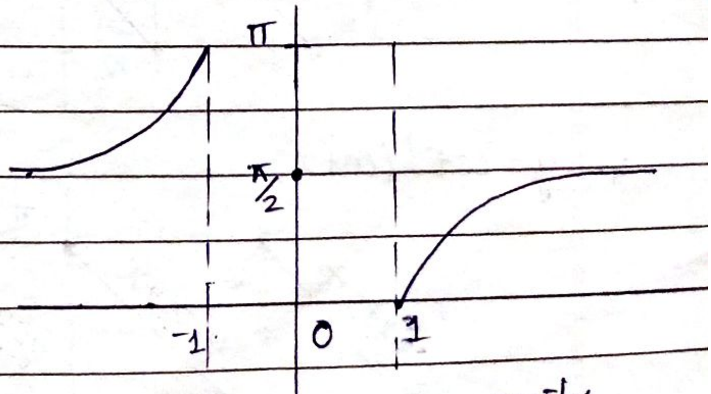


$y = \cot^{-1} x$.

5. $\sec x$.

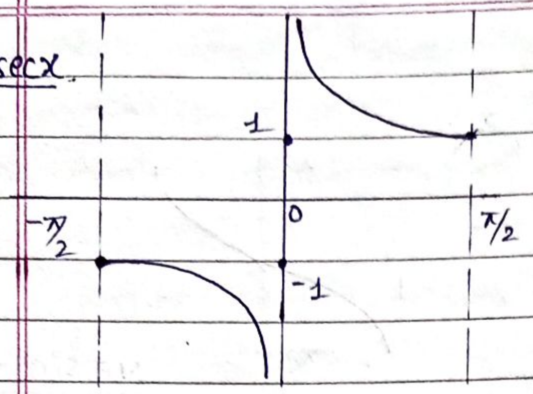


$y = \sec x$

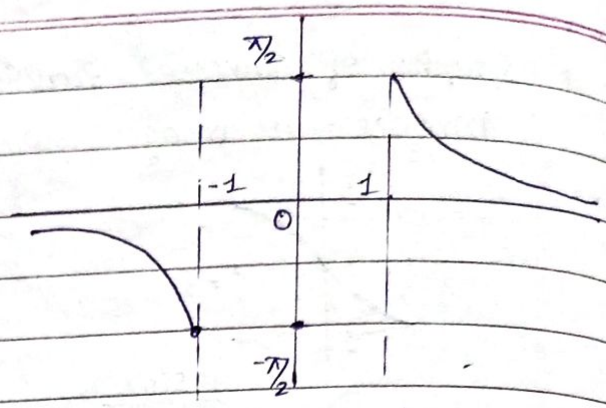


$y = \sec^{-1} x$

6· cosec x



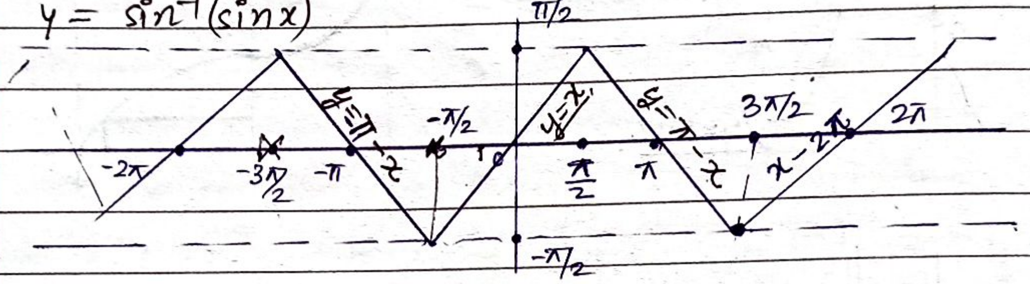
$y = \operatorname{cosec} x$



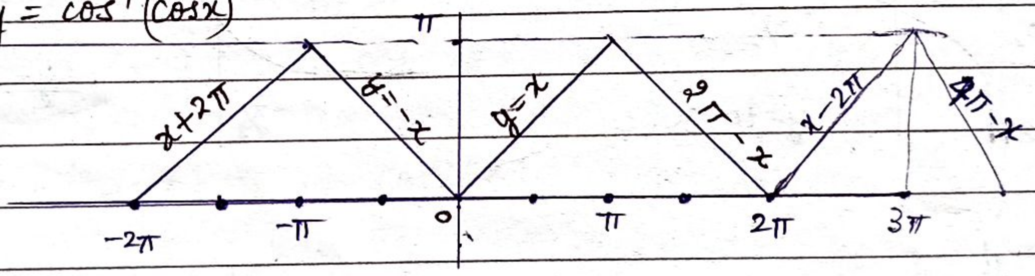
$y = \operatorname{cosec}^{-1} x$

* SPECIAL GRAPHS:

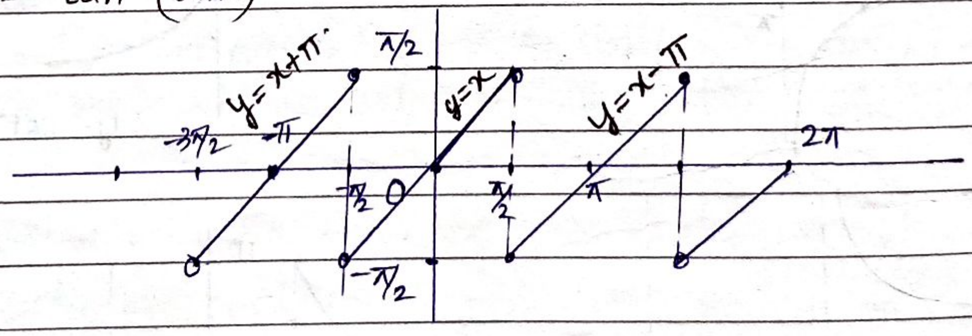
1. $y = \sin^{-1}(\sin x)$



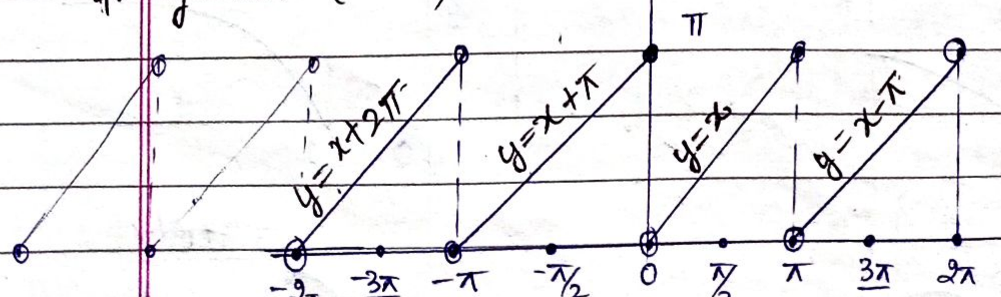
2. $y = \cos^{-1}(\cos x)$



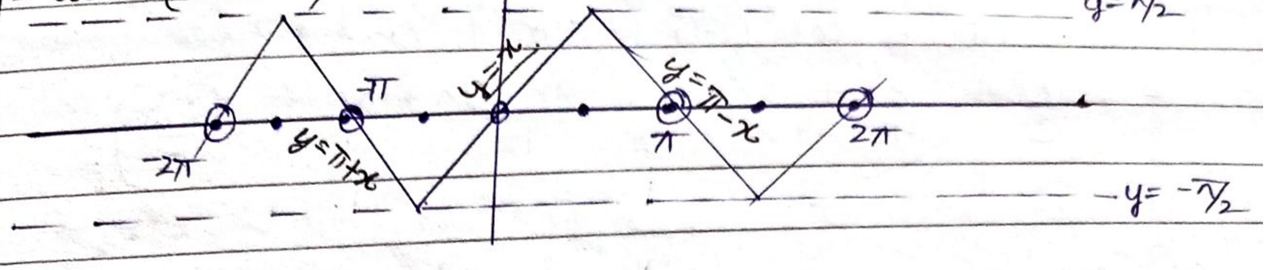
3. $y = \tan^{-1}(\tan x)$



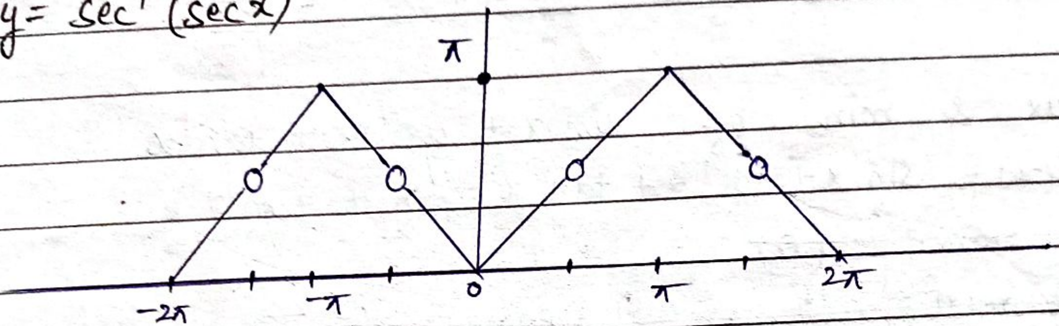
4. $y = \cot^{-1}(\cot x)$



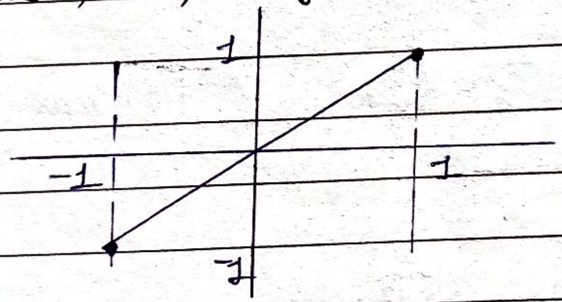
5. $y = \text{cosec}^{-1}(\text{cosec}\, x)$



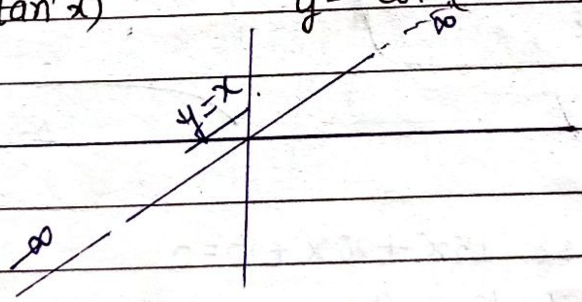
6. $y = \sec^{-1}(\sec x)$



7. $y = \cos(\cos^{-1} x)$, $y = \sin(\sin^{-1} x)$ {same graphs}



8. $y = \tan(\tan^{-1} x)$ $y = \cot(\cot^{-1} x)$ {same}



9. $y = \text{cosec}(\text{cosec}^{-1} x)$ $y = \sec(\sec^{-1} x)$ {same)