⇒ equation reducible to linear differential eqn.

Bernoulli's equation

$$f'(y)\frac{dy}{dx} + f(y)\,P(x) = Q(x)$$

$$f(y) = t. \qquad f'(y)\frac{dy}{dx} = \frac{dt}{dx}.$$

$$\frac{dt}{dx} + P(x)\,t = Q(x).$$

Q Solve: $$\frac{dy}{dx} - y\tan x = -y^2\sec x.$$

$$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}\tan x = -\sec x.$$

$$\frac{1}{y} = t \qquad \frac{-1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$$

$$+\frac{dt}{dx} + t\tan x = +\sec x$$

$$\frac{1}{y}\sec x = \tan x + c$$

$$t\sec x = \int \sec^2 x + c$$

$$\boxed{\sec x = y(\tan x + c)}$$

Q $$\sin y\frac{dy}{dx} = \cos y\,(1 - x\cos y).$$

$$\sin y\frac{dy}{dx} + x\cos^2 y = \cos y$$

$$\cos y = t \qquad -\sin y\frac{dy}{dx} = \frac{dt}{dx}$$

$$+\frac{dt}{dx} + t x = 1$$

$$+c\,e^{-\frac{x^2}{2}}$$

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$$\sin y \frac{dy}{dx} - \cos y = -\cos^2 y\, x.$$

$$\tan y \sec y \frac{dy}{dx} - \sec y = -x$$

$$\sec y = t$$

$$\frac{dt}{dx} - t = -x.$$

$$te^{-x} = -\left((-x)e^{-x} + e^{-x}\right) + c$$

$$\boxed{\sec y = (x-1) + ce^{x}}$$

Q solve: $\frac{dy}{dx}\left(xy + x^2y^3\right) = 1$

$$xy + x^2y^3 = \frac{dx}{dy}.$$

$$\frac{1}{x^2}\frac{dx}{dy} - \frac{xy}{x^2} = \cancel{x^2}y^2$$

$$\frac{1}{x} = t. \qquad -\frac{1}{x^2}\frac{dx}{dy} = \frac{dt}{dx}$$

$$\frac{dt}{dx} + yt = -y^3$$

$$te^{\frac{y^2}{2}} = \int -y^3 e^{\frac{y^2}{2}} + c.$$

$$\frac{e^{y^2/2}}{x} = \int -y^2 e^{t}.$$

$$\frac{y^2}{2} = t$$

$$y = \sqrt{2t}$$

$$\frac{e^{y^2/2}}{x} = -2\int 2t\, e^{t} + c$$

Q. $2x^3y\,dy + (1-y^2)(x^2y^2+y^2-1)\,dx = 0.$

$$2x^3y\frac{dy}{dx} = (y^2-1)(x^2y^2+y^2-1)$$

$$\frac{y}{(y^2-1)^2}\frac{dy}{dx} = \frac{y^2}{2x} + \frac{(y^2-1)}{2x^3}$$

$$\frac{2y}{(y^2-1)^2}\frac{dy}{dx} - \frac{y^2}{x(y^2-1)} = \frac{1}{x^3}$$

$$\frac{y^2}{(y^2-1)} = t$$

$$\frac{2y(y^2-1) - 2y(y^2)}{(y^2-1)^2} \qquad \frac{\cancel{2y^3} - 2y - \cancel{2y^3}}{(y^2-1)^2}$$

$$\frac{-2y}{(y^2-1)^2}\frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dt}{dx} + \frac{1}{x}t = -\frac{1}{x^3}$$

$$tx = \int \frac{-1}{x^2} + c$$

$$t = \frac{1}{x^2} + \frac{c}{x} \qquad \boxed{\frac{y^2}{(y^2-1)} = \frac{1}{x^2} + \frac{c}{x}}$$

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Q $\dfrac{dy}{dx} = \dfrac{1}{xy(x^2 \sin y^2 + 1)}$

$xy(x^2 \sin y^2 + 1) = \dfrac{dx}{dy}$.

$y x^2 \sin(y^2) + y = \dfrac{1}{x}\dfrac{dx}{dy}$

$-\dfrac{2}{x^3}\dfrac{dx}{dy} + \dfrac{2y}{x^2} = -2y \sin(y^2)$

$\dfrac{1}{x^2} = t \quad \dfrac{dx}{dy}\dfrac{-2}{x^3} = \dfrac{dt}{dy}$

$\dfrac{dt}{dy} + \dfrac{2y\,t}{1} = -2y \sin(y^2)$

$t e^{y^2} = -\int 2y \sin(y^2)\, e^{y^2}. + c$

$t e^{y^2} = -\int \sin(t)\, e^t + c.$

$\dfrac{e^{y^2}}{x^2} = -\dfrac{\sin y^2 e^{y^2} + \cos y^2 e^{y^2}}{2} + c$ $\quad$ $+\sin t \searrow e^t$, $-\cos t \searrow e^t$, $-\sin x \; e^t$

$$\boxed{\dfrac{1}{x^2} = \dfrac{\cos y^2 - \sin y^2}{2} + c e^{-y^2}}$$

$$\frac{dy}{dx} = 1 - x(y-x) - x^3(y-x)^3$$

$$(y-x) = t.$$

$$\frac{dy}{dx} - 1 = \frac{dt}{dx} \qquad \cancel{1+}\frac{dt}{dx} = \cancel{1} - tx - x^3t^3$$

$$\frac{dt}{dx} + tx = -x^3t^3$$

$$\frac{-1}{t^3}\frac{dt}{dx} + \frac{x}{t^2} = -x^3.$$

$\Rightarrow$ **General form of Variable Seperable.**

① $d(xy) = x\,dy + y\,dx.$

② $d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$

③ $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$

④ $d\left(x^2+y^2\right) = 2x\,dx + 2y\,dy.$

⑤ $d\left(\ln \frac{x}{y}\right) = \frac{y}{x}\left(\frac{y\,dx - x\,dy}{y^2}\right)$

⑥ $d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = \frac{d\left(\frac{y}{x}\right)}{1+\left(\frac{y^2}{x^2}\right)} = \frac{x\,dy - y\,dx}{x^2+y^2}$

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Q $y\,dx - x\,dy + x^3y^2\,dx = 0$

$$\frac{x\,dy - y\,dx}{y^2} = x^3 \cancel{y^2}\,dx$$

$$-d\left(\frac{x}{y}\right) = x^3\,dx$$

$$-\frac{x}{y} = \frac{x^4}{4} + c$$

$$\boxed{\frac{yx^4}{4} + cy + x = 0}$$

Q $x\,dy + y\,dx - \sqrt{1-x^2y^2}\,dx = 0.$

$$\frac{d(xy)}{\sqrt{1-x^2y^2}} = dx \qquad \boxed{xy = \sin(x+c)}$$

$$\sin^{-1}(xy) = x + c$$

Q $x\,dy - y\,dx - \sqrt{x^2-y^2}\,dx = 0$

$$\frac{x\,dy - y\,dx}{x^2} = \frac{\sqrt{x^2-y^2}\,dx}{x^2}.$$

$$\frac{x^2}{\sqrt{x^2-y^2}}\; d\left(\frac{y}{x}\right)$$

$$\frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}}\, d\left(\frac{y}{x}\right) = \frac{dx}{x}$$

$$\sin^{-1}\left(\frac{y}{x}\right) = \ln xc$$

$$\boxed{y = x\sin(\ln xc)}$$

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Q $\left(\frac{1}{x} - \frac{y^2}{(x-y)^2}\right) dx + \left(\frac{x^2}{(x-y)^2} - \frac{1}{y}\right) dy = 0$

$\frac{dx}{x} - \frac{dy}{y} = \frac{y^2}{(x-y)^2} dx - \frac{x^2 dy}{(x-y)^2}$

$\frac{dx}{x} - \frac{dy}{y} = \frac{x^2 y^2}{(x-y)^2}\left(\frac{dx}{x^2} - \frac{x^2 dy}{y^2}\right)$

$d\left(-\frac{1}{x} + \frac{1}{y}\right)$

$\left(\frac{y\,dx - x\,dy}{y^2}\right) = \frac{x^3 y}{(x-y)^2}\, d\left(\frac{1}{y} - \frac{1}{x}\right)$

$\frac{y}{x}\, d\left(\frac{x}{y}\right) = \frac{x^2 y^2}{(x-y)^2}\, d\left(\frac{x-y}{xy}\right)$

$\ln\left(\frac{x}{y}\right) = -\left(\frac{xy}{(x-y)}\right) + c$

$$\boxed{\ln\left(\frac{x}{y}\right) + \frac{xy}{(x-y)} = c}$$

Q $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$

$3x^2y^4\,dx + 2xy\,dx + 2x^3y^3\,dy - x^2dy = 0$

$\frac{3x^2y^4\,dx + 2x^3y^3\,dy}{y^2} = \frac{x^2dy - 2xy\,dx}{y^2}$

$d(x^3y^2) = \frac{x^2}{y^2}dy - \frac{2x}{y}dx$

$d(x^3y^2) = \underbrace{x^2 dy - 2xy\,dx}_{d(xy)}$

$$d(x^3y^2) = -d\left(\frac{x^2}{y}\right)$$

$$\boxed{x^3y^2 + \frac{x^2}{y} = c}$$

⇒ Orthogonal trajectories

Any curve, which cut every member of the given family of curves at right angle is called orthogonal trajectory of family.

for ex: $y = mx$ is orthogonal trajectory of family $x^2 + y^2 = r^2$.

F_1 family

Step (I) form a DE of F_1

Step (II) $\frac{dy}{dx} \longrightarrow -\frac{dx}{dy}$.

Step (III) solve the DE

F_2 family.

Q Find the orthogonal trajectory of $y^2 = ax$

$$2y\frac{dy}{dx} = a = \frac{y^2}{x}.$$

$$-2y\frac{dx}{dy} = \frac{y^2}{x}$$

$$-2x\,dx = y\,dy.$$

$$-x^2 = y^2 + c$$

$$\boxed{\frac{y^2}{2} + x^2 = c}$$ ellipse. and $\boxed{y=0}$.

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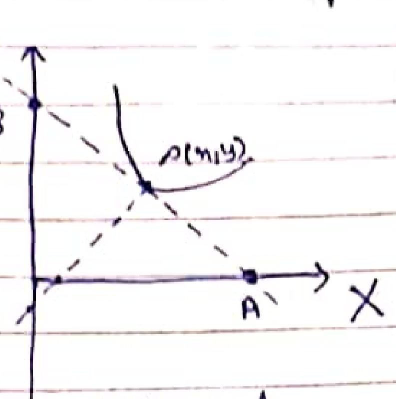
⇒ Geometrical problems.



eqn of tangent

$$(Y-y) = \frac{dy}{dx}(X-x)$$

$$(Y-y) = m(X-x)$$

$$m \to \frac{dy}{dx}$$

eqn of Normal.

$$(Y-y) = -\frac{dx}{dy}(X-x)$$

$$(Y-y) = -\frac{1}{m}(X-x) \qquad m \to \frac{dy}{dx}$$

Q find eqn of curve in which the portion of the tangent included btw ~~tangent~~ coordinate axis is bisected by the point of contact.

$$\cancel{y} - mx = \cancel{2}y.$$

$$\frac{1}{y}\frac{-dy}{dx} = \frac{1}{x}. \qquad \boxed{xy = c}$$

$$-\ln y = \ln xc.$$

Q find the eqn of the curve passing through $(e,-e)$ and which is such that the portion of normal at any point and x axis is bisected by the line $x+2y=$

$$(Y-y) = \frac{-1}{m}(X-x)$$

$$(x+my, 0) \qquad (x, y)$$

$$\left(\frac{2x+my}{2}, \frac{y}{2}\right) \qquad x+2y=0$$

$$\frac{2x+my}{2} + \frac{2y}{2} = 0$$

$$2x + 2y + \frac{dy}{dx}y = 0.$$

$$2\frac{x}{y} + 2 = -\frac{dy}{dx} \qquad v = \frac{x}{y}$$

$$-(2v+2) = v + y\frac{dv}{dy}$$

$$-3v-2 = y\frac{dv}{dy}$$

$$-\frac{dy}{y} = \frac{dv}{(3v+2)} \qquad c = 3xy^2 + 2y^3 \quad (e, -e)$$

$$\ln\frac{c}{y} = \frac{\ln(3v+2)}{3} \qquad \boxed{e^3 = 3xy^2 + 2y^3}$$

$$\frac{c}{y^3} = 3\frac{x}{y} + 2$$

⇒ Word problems.

Q Suppose the growth of population is proportional to the no present. It population of colony double in 25 yea[rs] find in how many years the population becomes triple.

let population no be denoted by n, & let at time t_0 population be n_0

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$$\frac{dn}{dt} = kn$$

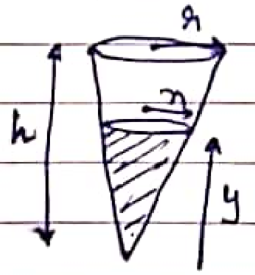
$$\ln n \Big|_{n_0}^{2n_0} = kt \Big|_{t_0}^{25+t_0}$$

$$\frac{\ln 2}{(25 \cancel{-t_0})} = k$$

$$\ln n \Big|_{n_0}^{3n_0} = \frac{\ln 2}{25}(t)$$

$$\boxed{t = \frac{\ln 3 (25)}{(\ln 2)\cancel{25}}}$$

Q A right circular cone of radius R and height h contains a liquid which evaporates proportional to area in contact to air (k > 0) find time after which cone become empty.



$$\frac{dv}{dt} = k\pi r^2$$

$$\frac{d\left(\frac{\pi r^2 y}{3}\right)}{dt} = -k\pi r^2$$

$$\frac{y}{r} = \frac{h}{R}$$

$$\boxed{y = \frac{hr}{R}}$$

$$d\left(\frac{\pi r^3}{3}\frac{h}{R}\right) = -k\pi r^2 dt$$

$$\cancel{\pi r^2}\frac{h}{R}\, dr \Big|_R^0 = -k\cancel{\pi r^2}\, dt \Big|_0^t$$

$$-\frac{h}{R}R = -kt$$

$$\boxed{t = \frac{h}{k}}$$ Ans