or $\sin v = \log|x| + \log|C|$

or $\sin v = \log|Cx|$

Replacing v by $\frac{y}{x}$, we get

$$\sin\left(\frac{y}{x}\right) = \log|Cx|$$

which is the general solution of the differential equation (1).

Example 17 Show that the differential equation $2y\,e^{\frac{x}{y}}dx + \left(y - 2x\,e^{\frac{x}{y}}\right)dy = 0$ is homogeneous and find its particular solution, given that, $x = 0$ when $y = 1$.

Solution The given differential equation can be written as

$$\frac{dx}{dy} = \frac{2x\,e^{\frac{x}{y}} - y}{2y\,e^{\frac{x}{y}}} \quad \text{... (1)}$$

Let $$F(x, y) = \frac{2xe^{\frac{x}{y}} - y}{2ye^{\frac{x}{y}}}$$

Then $$F(\lambda x, \lambda y) = \frac{\lambda\left(2xe^{\frac{x}{y}} - y\right)}{\lambda\left(2ye^{\frac{x}{y}}\right)} = \lambda^0[F(x, y)]$$

Thus, $F(x, y)$ is a homogeneous function of degree zero. Therefore, the given differential equation is a homogeneous differential equation.

To solve it, we make the substitution

$$x = vy \quad \text{... (2)}$$

Differentiating equation (2) with respect to y, we get

$$\frac{dx}{dy} = v + y\frac{dv}{dy}$$

Substituting the value of x and $\frac{dx}{dy}$ in equation (1), we get

$$v + y\frac{dv}{dy} = \frac{2v\,e^v - 1}{2e^v}$$

or $$y\frac{dv}{dy} = \frac{2v\,e^v - 1}{2e^v} - v$$

or $$y\frac{dv}{dy} = -\frac{1}{2e^v}$$

or $$2e^v\,dv = \frac{-dy}{y}$$

or $$\int 2e^v \cdot dv = -\int \frac{dy}{y}$$

or $$2\,e^v = -\log|y| + C$$

and replacing v by $\frac{x}{y}$, we get

$$2\,e^{\frac{x}{y}} + \log|y| = C \qquad \text{... (3)}$$

Substituting $x = 0$ and $y = 1$ in equation (3), we get

$$2\,e^0 + \log|1| = C \Rightarrow C = 2$$

Substituting the value of C in equation (3), we get

$$2\,e^{\frac{x}{y}} + \log|y| = 2$$

which is the particular solution of the given differential equation.

Example 18 Show that the family of curves for which the slope of the tangent at any point (x, y) on it is $\frac{x^2 + y^2}{2xy}$, is given by $x^2 - y^2 = cx$.

Solution We know that the slope of the tangent at any point on a curve is $\frac{dy}{dx}$.

Therefore, $$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$$

or $$\frac{dy}{dx} = \frac{1+\frac{y^2}{x^2}}{\frac{2y}{x}} \qquad \ldots (1)$$

Clearly, (1) is a homogenous differential equation. To solve it we make substitution

$$y = vx$$

Differentiating $y = vx$ with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

or $$v + x\frac{dv}{dx} = \frac{1+v^2}{2v}$$

or $$x\frac{dv}{dx} = \frac{1-v^2}{2v}$$

$$\frac{2v}{1-v^2}dv = \frac{dx}{x}$$

or $$\frac{2v}{v^2-1}dv = -\frac{dx}{x}$$

Therefore $$\int\frac{2v}{v^2-1}dv = -\int\frac{1}{x}dx$$

or $$\log|v^2 - 1| = -\log|x| + \log|C_1|$$

or $$\log|(v^2 - 1)(x)| = \log|C_1|$$

or $$(v^2 - 1)x = \pm C_1$$

Replacing v by $\frac{y}{x}$, we get

$$\left(\frac{y^2}{x^2}-1\right)x = \pm C_1$$

or $$(y^2 - x^2) = \pm C_1 x \text{ or } x^2 - y^2 = Cx$$

EXERCISE 9.5

In each of the Exercises 1 to 10, show that the given differential equation is homogeneous and solve each of them.

1. $(x^2 + xy)\, dy = (x^2 + y^2)\, dx$

2. $y' = \dfrac{x+y}{x}$

3. $(x - y)\, dy - (x + y)\, dx = 0$

4. $(x^2 - y^2)\, dx + 2xy\, dy = 0$

5. $x^2 \dfrac{dy}{dx} = x^2 - 2y^2 + xy$

6. $x\, dy - y\, dx = \sqrt{x^2 + y^2}\, dx$

7. $\left\{x\cos\left(\dfrac{y}{x}\right) + y\sin\left(\dfrac{y}{x}\right)\right\} y\, dx = \left\{y\sin\left(\dfrac{y}{x}\right) - x\cos\left(\dfrac{y}{x}\right)\right\} x\, dy$

8. $x\dfrac{dy}{dx} - y + x\sin\left(\dfrac{y}{x}\right) = 0$

9. $y\, dx + x\log\left(\dfrac{y}{x}\right) dy - 2x\, dy = 0$

10. $1 \quad e^{\frac{x}{y}} \quad dx \quad e^{\frac{x}{y}} \quad 1 \quad \dfrac{x}{y} \quad dy \quad 0$

For each of the differential equations in Exercises from 11 to 15, find the particular solution satisfying the given condition:

11. $(x + y)\, dy + (x - y)\, dx = 0;\ y = 1$ when $x = 1$

12. $x^2\, dy + (xy + y^2)\, dx = 0;\ y = 1$ when $x = 1$

13. $x\sin^2 \dfrac{y}{x} \quad y \quad dx \quad x\, dy \quad 0;\ y \quad \dfrac{\;}{4}$ when $x = 1$

14. $\dfrac{dy}{dx} - \dfrac{y}{x} + \operatorname{cosec}\left(\dfrac{y}{x}\right) = 0;\ \ y = 0$ when $x = 1$

15. $2xy + y^2 - 2x^2\dfrac{dy}{dx} = 0;\ \ y = 2$ when $x = 1$

16. A homogeneous differential equation of the from $\dfrac{dx}{dy} = h\left(\dfrac{x}{y}\right)$ can be solved by making the substitution.

(A) $y = vx$ (B) $v = yx$ (C) $x = vy$ (D) $x = v$

17. Which of the following is a homogeneous differential equation?

(A) $(4x + 6y + 5)\, dy - (3y + 2x + 4)\, dx = 0$

(B) $(xy)\, dx - (x^3 + y^3)\, dy = 0$

(C) $(x^3 + 2y^2)\, dx + 2xy\, dy = 0$

(D) $y^2\, dx + (x^2 - xy - y^2)\, dy = 0$

9.5.3 *Linear differential equations*

A differential equation of the from

$$\frac{dy}{dx} + \mathrm{P}y = \mathrm{Q}$$

where, P and Q are constants or functions of x only, is known as a first order linear differential equation. Some examples of the first order linear differential equation are

$$\frac{dy}{dx} + y = \sin x$$

$$\frac{dy}{dx} + \left(\frac{1}{x}\right) y = e^x$$

$$\frac{dy}{dx} + \left(\frac{y}{x \log x}\right) = \frac{1}{x}$$

Another form of first order linear differential equation is

$$\frac{dx}{dy} + \mathrm{P}_1 x = \mathrm{Q}_1$$

where, P_1 and Q_1 are constants or functions of y only. Some examples of this type of differential equation are

$$\frac{dx}{dy} + x = \cos y$$

$$\frac{dx}{dy} + \frac{-2x}{y} = y^2 e^{-y}$$

To solve the first order linear differential equation of the type

$$\frac{dy}{dx} \quad \mathrm{P}y = \mathrm{Q} \qquad \text{... (1)}$$

Multiply both sides of the equation by a function of x say $g(x)$ to get

$$g(x) \frac{dy}{dx} + \mathrm{P}.(g(x))\, y = \mathrm{Q}\, .\, g(x) \qquad \text{... (2)}$$

Choose $g(x)$ in such a way that R.H.S. becomes a derivative of $y \,.\, g(x)$.

i.e. $$g(x)\frac{dy}{dx} + \text{P}.g(x)\,y = \frac{d}{dx}[y\,.\,g(x)]$$

or $$g(x)\frac{dy}{dx} + \text{P}.g(x)\,y = g(x)\frac{dy}{dx} + y\,g'(x)$$

$\Rightarrow$ $$\text{P}.g(x) = g'(x)$$

or $$\text{P} = \frac{g'(x)}{g(x)}$$

Integrating both sides with respect to x, we get

$$\int \text{P}dx = \int \frac{g'(x)}{g(x)}\,dx$$

or $$\int \text{P}\cdot dx = \log(g(x))$$

or $$g(x) = e^{\int \text{P}\,dx}$$

On multiplying the equation (1) by $g(x) = e^{\int \text{P}\,dx}$, the L.H.S. becomes the derivative of some function of x and y. This function $g(x) = e^{\int \text{P}\,dx}$ is called *Integrating Factor* (I.F.) of the given differential equation.

Substituting the value of $g(x)$ in equation (2), we get

$$e^{\int \text{P}\,dx}\frac{dy}{dx} + \text{P}e^{\int \text{P}\,dx}\,y = \text{Q}\,e^{\int \text{P}\,dx}$$

or $$\frac{d}{dx}\left(y\,e^{\int \text{P}dx}\right) = \text{Q}e^{\int \text{P}dx}$$

Integrating both sides with respect to x, we get

$$y\,e^{\int \text{P}dx} = \int\left(\text{Q}\,e^{\int \text{P}dx}\right)dx$$

or $$y = e^{-\int \text{P}dx}\cdot\int\left(\text{Q}\,e^{\int \text{P}dx}\right)dx + \text{C}$$

which is the general solution of the differential equation.

Steps involved to solve first order linear differential equation:

(i) Write the given differential equation in the form $\frac{dy}{dx} + \text{P}y = \text{Q}$ where P, Q are constants or functions of x only.

(ii) Find the Integrating Factor (I.F) = $e^{\ \text{P}\,dx}$.

(iii) Write the solution of the given differential equation as

$$y \text{ (I.F)} = \quad \text{Q} \times \text{I.F} \ \ dx \quad \text{C}$$

In case, the first order linear differential equation is in the form $\frac{dx}{dy} + \text{P}_1 x = \text{Q}_1$, where, P_1 and Q_1 are constants or functions of y only. Then I.F = $e^{\ \text{P}_1\,dy}$ and the solution of the differential equation is given by

$$x \text{ . (I.F)} = \int (\text{Q}_1 \times \text{I.F})\, dy + \text{C}$$

Example 19 Find the general solution of the differential equation $\frac{dy}{dx} - y = \cos x$.

Solution Given differential equation is of the form

$$\frac{dy}{dx} + \text{P}y = \text{Q}, \text{ where P} = -1 \text{ and Q} = \cos x$$

Therefore $\qquad \text{I.F} = e^{\ 1\,dx} \quad e^{\ x}$

Multiplying both sides of equation by I.F, we get

$$e^{-x}\frac{dy}{dx} - e^{-x}y = e^{-x}\cos x$$

or $$\frac{dy}{dx}\left(ye^{-x}\right) = e^{-x}\cos x$$

On integrating both sides with respect to x, we get

$$ye^{-x} = \int e^{-x}\cos x\, dx + \text{C} \qquad \text{... (1)}$$

Let $$\text{I} = \int e^{-x}\cos x\, dx$$

$$= \cos x\left(\frac{e^{-x}}{-1}\right) - \int(-\sin x)\,(-e^{-x})\, dx$$

$$= -\cos x\, e^{-x} - \int \sin x\, e^{-x}\, dx$$

$$= -\cos x\, e^{-x} - \left[\sin x(-e^{-x}) - \int \cos x\,(-e^{-x})\, dx\right]$$

$$= -\cos x\, e^{-x} + \sin x\, e^{-x} - \int \cos x\, e^{-x}\, dx$$

or $$I = -\, e^{-x} \cos x + \sin x\, e^{-x} - I$$

or $$2I = (\sin x - \cos x)\, e^{-x}$$

or $$I = \frac{(\sin x - \cos x) e^{-x}}{2}$$

Substituting the value of I in equation (1), we get

$$ye^{-x} = \left(\frac{\sin x - \cos x}{2}\right)e^{-x} + C$$

or $$y = \left(\frac{\sin x - \cos x}{2}\right) + C e^{x}$$

which is the general solution of the given differential equation.

Example 20 Find the general solution of the differential equation $x\frac{dy}{dx} + 2y = x^2\ (x \neq 0)$.

Solution The given differential equation is

$$x\frac{dy}{dx} + 2y = x^2 \qquad \ldots (1)$$

Dividing both sides of equation (1) by x, we get

$$\frac{dy}{dx} + \frac{2}{x}y = x$$

which is a linear differential equation of the type $\frac{dy}{dx} + Py = Q$, where $P = \frac{2}{x}$ and $Q = x$.

So $$\text{I.F} = e^{\int \frac{2}{x} dx} = e^{2 \log x} = e^{\log x^2} = x^2 \ [as\ e^{\log f(x)} = f(x)]$$

Therefore, solution of the given equation is given by

$$y \cdot x^2 = \int (x)(x^2)\, dx + C = \int x^3 dx + C$$

or $$y = \frac{x^2}{4} + C x^{-2}$$

which is the general solution of the given differential equation.

Example 21 Find the general solution of the differential equation $y\,dx - (x + 2y^2)\,dy = 0$.

Solution The given differential equation can be written as

$$\frac{dx}{dy} - \frac{x}{y} = 2y$$

This is a linear differential equation of the type $\frac{dx}{dy} + P_1 x = Q_1$, where $P_1 = -\frac{1}{y}$ and $Q_1 = 2y$. Therefore $\text{I.F} = e^{\int -\frac{1}{y}dy} = e^{-\log y} = e^{\log (y)^{-1}} = \frac{1}{y}$

Hence, the solution of the given differential equation is

$$x\frac{1}{y} = \int (2y)\left(\frac{1}{y}\right)dy + C$$

or $$\frac{x}{y} = \int (2dy) + C$$

or $$\frac{x}{y} = 2y + C$$

or $$x = 2y^2 + Cy$$

which is a general solution of the given differential equation.

Example 22 Find the particular solution of the differential equation

$$\frac{dy}{dx} + y\cot x = 2x + x^2 \cot x \ (x \neq 0)$$

given that $y = 0$ when $x = \frac{\pi}{2}$.

Solution The given equation is a linear differential equation of the type $\frac{dy}{dx} + Py = Q$, where $P = \cot x$ and $Q = 2x + x^2 \cot x$. Therefore

$$\text{I.F} = e^{\cot x\, dx} \quad e^{\log \sin x} \quad \sin x$$

Hence, the solution of the differential equation is given by

$$y \,.\, \sin x = \int (2x + x^2 \cot x) \sin x\, dx + C$$

or $$y \sin x = \int 2x \sin x \, dx + \int x^2 \cos x \, dx + C$$

or $$y \sin x = \sin x \left(\frac{2x^2}{2}\right) - \int \cos x \left(\frac{2x^2}{2}\right) dx + \int x^2 \cos x \, dx + C$$

or $$y \sin x = x^2 \sin x - \int x^2 \cos x \, dx + \int x^2 \cos x \, dx + C$$

or $$y \sin x = x^2 \sin x + C \quad \ldots (1)$$

Substituting $y = 0$ and $x = \frac{\pi}{2}$ in equation (1), we get

$$0 = \left(\frac{\pi}{2}\right)^2 \sin\left(\frac{\pi}{2}\right) + C$$

or $$C = \frac{-\pi^2}{4}$$

Substituting the value of C in equation (1), we get

$$y \sin x = x^2 \sin x - \frac{\pi^2}{4}$$

or $$y = x^2 - \frac{\pi^2}{4 \sin x} \quad (\sin x \neq 0)$$

which is the particular solution of the given differential equation.

Example 23 Find the equation of a curve passing through the point (0, 1). If the slope of the tangent to the curve at any point (x, y) is equal to the sum of the x coordinate (abscissa) and the product of the x coordinate and y coordinate (ordinate) of that point.

Solution We know that the slope of the tangent to the curve is $\frac{dy}{dx}$.

Therefore, $$\frac{dy}{dx} = x + xy$$

or $$\frac{dy}{dx} - xy = x \quad \ldots (1)$$

This is a linear differential equation of the type $\frac{dy}{dx} + Py = Q$, where $P = -x$ and $Q = x$.

Therefore, $$\text{I.F} = e^{\int -x \, dx} = e^{\frac{-x^2}{2}}$$

Hence, the solution of equation is given by

$$y \cdot e^{\frac{-x^2}{2}} = \int (x)\left(e^{\frac{-x^2}{2}}\right) dx + \text{C} \qquad \dots (2)$$

Let $$\text{I} = \int (x)\, e^{\frac{-x^2}{2}}\, dx$$

Let $\frac{-x^2}{2} = t$, then $-x\,dx = dt$ or $x\,dx = -dt$.

Therefore, $$\text{I} = -\int e^t\, dt = -e^t = -e^{\frac{-x^2}{2}}$$

Substituting the value of I in equation (2), we get

$$y\, e^{\frac{-x^2}{2}} = e^{\frac{x^2}{2}} + \text{C}$$

or $$y = -1 + \text{C}\, e^{\frac{x^2}{2}} \qquad \dots (3)$$

Now (3) represents the equation of family of curves. But we are interested in finding a particular member of the family passing through (0, 1). Substituting $x = 0$ and $y = 1$ in equation (3) we get

$$1 = -1 + \text{C} \cdot e^0 \quad \text{or} \quad \text{C} = 2$$

Substituting the value of C in equation (3), we get

$$y = -1 + 2\, e^{\frac{x^2}{2}}$$

which is the equation of the required curve.

EXERCISE 9.6

For each of the differential equations given in Exercises 1 to 12, find the general solution:

1. $\frac{dy}{dx} + 2y = \sin x$ **2.** $\frac{dy}{dx} + 3y = e^{-2x}$ **3.** $\frac{dy}{dx} + \frac{y}{x} = x^2$

4. $\frac{dy}{dx} + (\sec x)\, y = \tan x \left(0 \le x < \frac{\pi}{2}\right)$ **5.** $\cos^2 x \frac{dy}{dx} + y = \tan x \left(0 \le x < \frac{\pi}{2}\right)$

6. $x\frac{dy}{dx} + 2y = x^2 \log x$ **7.** $x \log x \frac{dy}{dx} + y = \frac{2}{x} \log x$

8. $(1 + x^2)\, dy + 2xy\, dx = \cot x\, dx \ (x \ne 0)$

9. $x\frac{dy}{dx} + y - x + xy\cot x = 0 \ (x \neq 0)$ **10.** $(x + y)\frac{dy}{dx} = 1$

11. $y\,dx + (x - y^2)\,dy = 0$ **12.** $(x + 3y^2)\frac{dy}{dx} = y \ (y > 0)$.

For each of the differential equations given in Exercises 13 to 15, find a particular solution satisfying the given condition:

13. $\frac{dy}{dx} + 2y\tan x = \sin x;\ y = 0$ when $x = \frac{\pi}{3}$

14. $(1 + x^2)\frac{dy}{dx} + 2xy = \frac{1}{1 + x^2};\ y = 0$ when $x = 1$

15. $\frac{dy}{dx} - 3y\cot x = \sin 2x;\ y = 2$ when $x = \frac{\pi}{2}$

16. Find the equation of a curve passing through the origin given that the slope of the tangent to the curve at any point (x, y) is equal to the sum of the coordinates of the point.

17. Find the equation of a curve passing through the point (0, 2) given that the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at that point by 5.

18. The Integrating Factor of the differential equation $x\frac{dy}{dx} - y = 2x^2$ is

(A) e^{-x} (B) e^{-y} (C) $\frac{1}{x}$ (D) x

19. The Integrating Factor of the differential equation

$(1 - y^2)\frac{dx}{dy} + yx = ay(\ 1\ \ y\ \ 1)$ is

(A) $\frac{1}{y^2\ \ 1}$ (B) $\frac{1}{\sqrt{y^2 - 1}}$ (C) $\frac{1}{1 - y^2}$ (D) $\frac{1}{\sqrt{1 - y^2}}$

Miscellaneous Examples

Example 24 Verify that the function $y = c_1 e^{ax}\cos bx + c_2 e^{ax}\sin bx$, where c_1, c_2 are arbitrary constants is a solution of the differential equation

$$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0$$

Solution The given function is

$$y = e^{ax}\,[c_1 \cos bx + c_2 \sin bx] \qquad \text{... (1)}$$

Differentiating both sides of equation (1) with respect to *x*, we get

$$\frac{dy}{dx} = e^{ax} \quad -bc_1 \sin bx \quad bc_2 \cos bx \qquad c_1 \cos bx \quad c_2 \sin bx \;\, e^{ax} \quad a$$

or $$\frac{dy}{dx} = e^{ax}[(bc_2 + ac_1)\cos bx + (ac_2 - bc_1)\sin bx] \qquad \text{... (2)}$$

Differentiating both sides of equation (2) with respect to *x*, we get

$$\frac{d^2y}{dx^2} = e^{ax}[(bc_2 \quad ac_1)(\quad b\sin bx) \quad (ac_2 \quad bc_1)(b\cos bx)]$$

$$+\ [(bc_2 + ac_1)\cos bx + (ac_2 - bc_1)\sin bx]\,e^{ax}.a$$

$$= e^{ax}[(a^2c_2 - 2abc_1 - b^2c_2)\sin bx + (a^2c_1 + 2abc_2 - b^2c_1)\cos bx]$$

Substituting the values of $\frac{d^2y}{dx^2}, \frac{dy}{dx}$ and *y* in the given differential equation, we get

$$\text{L.H.S.} = e^{ax}[a^2c_2 - 2abc_1 - b^2c_2)\sin bx + (a^2c_1 + 2abc_2 - b^2c_1)\cos bx]$$

$$2ae^{ax}[(bc_2 \quad ac_1)\cos bx \quad (ac_2 \quad bc_1)\sin bx]$$

$$(a^2 \quad b^2)e^{ax}[c_1\cos bx \quad c_2 \sin bx]$$

$$= e^{ax}\begin{bmatrix} \left(a^2c_2 - 2abc_1 - b^2c_2 - 2a^2c_2 + 2abc_1 + a^2c_2 + b^2c_2\right)\sin bx \\ +(a^2c_1 + 2abc_2 - b^2c_1 - 2abc_2 - 2a^2c_1 + a^2c_1 + b^2c_1)\cos bx \end{bmatrix}$$

$$= e^{ax}[0 \times \sin bx + 0\cos bx] = e^{ax} \times 0 = 0 \ = \text{R.H.S.}$$

Hence, the given function is a solution of the given differential equation.

Example 25 Form the differential equation of the family of circles in the second quadrant and touching the coordinate axes.

Solution Let C denote the family of circles in the second quadrant and touching the coordinate axes. Let (–*a*, *a*) be the coordinate of the centre of any member of this family (see Fig 9.6).

Equation representing the family C is

$$(x + a)^2 + (y - a)^2 = a^2 \quad \text{... (1)}$$

or $$x^2 + y^2 + 2ax - 2ay + a^2 = 0 \quad \text{... (2)}$$

Differentiating equation (2) with respect to x, we get

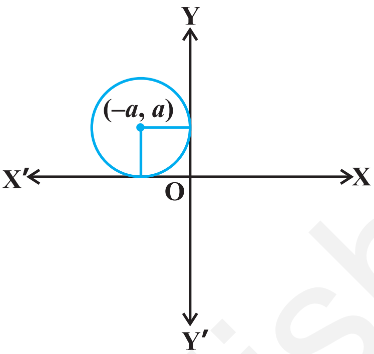

Fig 9.6

$$2x + 2y\frac{dy}{dx} + 2a - 2a\frac{dy}{dx} = 0$$

or $$x + y\frac{dy}{dx} = a\left(\frac{dy}{dx} - 1\right)$$

or $$a = \frac{x + y\,y'}{y' - 1}$$

Substituting the value of a in equation (1), we get

$$\left[x + \frac{x + y\,y'}{y' - 1}\right]^2 + \left[y - \frac{x + y\,y'}{y' - 1}\right]^2 = \left[\frac{x + y\,y'}{y' - 1}\right]^2$$

or $$[xy' - x + x + y\,y']^2 + [y\,y' - y - x - y\,y']^2 = [x + y\,y']^2$$

or $$(x + y)^2\, y'^2 + [x + y]^2 = [x + y\,y']^2$$

or $$(x + y)^2\, [(y')^2 + 1] = [x + y\,y']^2$$

which is the differential equation representing the given family of circles.

Example 26 Find the particular solution of the differential equation $\log\left(\frac{dy}{dx}\right) = 3x + 4y$ given that $y = 0$ when $x = 0$.

Solution The given differential equation can be written as

$$\frac{dy}{dx} = e^{(3x + 4y)}$$

or $$\frac{dy}{dx} = e^{3x} \,.\, e^{4y} \quad \text{... (1)}$$

Separating the variables, we get

$$\frac{dy}{e^{4y}} = e^{3x}\, dx$$

Therefore $$\int e^{-4y} dy = \int e^{3x} dx$$

or $$\frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} + C$$

or $$4\,e^{3x} + 3\,e^{-4y} + 12\,C = 0 \qquad \text{... (2)}$$

Substituting $x = 0$ and $y = 0$ in (2), we get

$$4 + 3 + 12\,C = 0 \text{ or } C = \frac{-7}{12}$$

Substituting the value of C in equation (2), we get

$$4e^{3x} + 3e^{-4y} - 7 = 0,$$

which is a particular solution of the given differential equation.

Example 27 Solve the differential equation

$$(x\,dy - y\,dx)\,y\sin\left(\frac{y}{x}\right) = (y\,dx + x\,dy)\,x\cos\left(\frac{y}{x}\right).$$

Solution The given differential equation can be written as

$$\left[x\,y\sin\left(\frac{y}{x}\right) - x^2\cos\left(\frac{y}{x}\right)\right]dy = \left[xy\cos\left(\frac{y}{x}\right) + y^2\sin\left(\frac{y}{x}\right)\right]dx$$

or $$\frac{dy}{dx} = \frac{xy\cos\left(\frac{y}{x}\right) + y^2\sin\left(\frac{y}{x}\right)}{xy\sin\left(\frac{y}{x}\right) - x^2\cos\left(\frac{y}{x}\right)}$$

Dividing numerator and denominator on RHS by x^2, we get

$$\frac{dy}{dx} = \frac{\frac{y}{x}\cos\left(\frac{y}{x}\right) + \left(\frac{y^2}{x^2}\right)\sin\left(\frac{y}{x}\right)}{\frac{y}{x}\sin\left(\frac{y}{x}\right) - \cos\left(\frac{y}{x}\right)} \qquad \text{... (1)}$$

Clearly, equation (1) is a homogeneous differential equation of the form $\frac{dy}{dx} = g\left(\frac{y}{x}\right)$.

To solve it, we make the substitution

$$y = vx \qquad \text{... (2)}$$

or $$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

or $$v + x\frac{dv}{dx} = \frac{v\cos v + v^2 \sin v}{v\sin v - \cos v} \quad \text{(using (1) and (2))}$$

or $$x\frac{dv}{dx} = \frac{2v\cos v}{v\sin v - \cos v}$$

or $$\left(\frac{v\sin v - \cos v}{v\cos v}\right)dv = \frac{2\,dx}{x}$$

Therefore $$\int\left(\frac{v\sin v - \cos v}{v\cos v}\right)dv = 2\int\frac{1}{x}\,dx$$

or $$\int \tan v\,dv - \int\frac{1}{v}\,dv = 2\int\frac{1}{x}\,dx$$

or $$\log|\sec v| - \log|v| = 2\log|x| + \log|C_1|$$

or $$\log\left|\frac{\sec v}{v\,x^2}\right| = \log|C_1|$$

or $$\frac{\sec v}{v\,x^2} = \pm C_1 \quad \text{... (3)}$$

Replacing v by $\frac{y}{x}$ in equation (3), we get

$$\frac{\sec\left(\frac{y}{x}\right)}{\left(\frac{y}{x}\right)(x^2)} = C \text{ where, } C = \pm C_1$$

or $$\sec\left(\frac{y}{x}\right) = C\,xy$$

which is the general solution of the given differential equation.

Example 28 Solve the differential equation

$$(\tan^{-1}y - x)\,dy = (1 + y^2)\,dx.$$

Solution The given differential equation can be written as

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2} \quad \text{... (1)}$$