

18. $\frac{e^{\tan^{-1}x}}{1+x^2}$

19. $\frac{e^{2x}-1}{e^{2x}+1}$

20. $\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$

21. $\tan^2(2x-3)$

22. $\sec^2(7-4x)$

23. $\frac{\sin^{-1}x}{\sqrt{1-x^2}}$

24. $\frac{2\cos x-3\sin x}{6\cos x+4\sin x}$

25. $\frac{1}{\cos^2 x(1-\tan x)^2}$

26. $\frac{\cos\sqrt{x}}{\sqrt{x}}$

27. $\sqrt{\sin 2x} \cos 2x$

28. $\frac{\cos x}{\sqrt{1+\sin x}}$

29. $\cot x \log \sin x$

30. $\frac{\sin x}{1+\cos x}$

31. $\frac{\sin x}{(1+\cos x)^2}$

32. $\frac{1}{1+\cot x}$

33. $\frac{1}{1-\tan x}$

34. $\frac{\sqrt{\tan x}}{\sin x \cos x}$

35. $\frac{(1+\log x)^2}{x}$

36. $\frac{(x+1)(x+\log x)^2}{x}$

37. $\frac{x^3 \sin(\tan^{-1}x^4)}{1+x^8}$

Choose the correct answer in Exercises 38 and 39.

38. $\int \frac{10x^9 + 10^x \log_e 10 dx}{x^{10} + 10^x}$ equals

(A) $10^x - x^{10} + C$

(B) $10^x + x^{10} + C$

(C) $(10^x - x^{10})^{-1} + C$

(D) $\log(10^x + x^{10}) + C$

39. $\int \frac{dx}{\sin^2 x \cos^2 x}$ equals

(A) $\tan x + \cot x + C$

(B) $\tan x - \cot x + C$

(C) $\tan x \cot x + C$

(D) $\tan x - \cot 2x + C$

7.3.2 Integration using trigonometric identities

When the integrand involves some trigonometric functions, we use some known identities to find the integral as illustrated through the following example.

Example 7 Find (i) $\int \cos^2 x dx$ (ii) $\int \sin 2x \cos 3x dx$ (iii) $\int \sin^3 x dx$

Solution

- (i) Recall the identity
- $\cos 2x = 2 \cos^2 x - 1$
- , which gives

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\begin{aligned} \text{Therefore, } \int \cos^2 x \, dx &= \frac{1}{2} \int (1 + \cos 2x) \, dx = \frac{1}{2} \int dx + \frac{1}{2} \int \cos 2x \, dx \\ &= \frac{x}{2} + \frac{1}{4} \sin 2x + C \end{aligned}$$

- (ii) Recall the identity
- $\sin x \cos y = \frac{1}{2} [\sin(x+y) + \sin(x-y)]$
- (Why?)

$$\begin{aligned} \text{Then } \int \sin 2x \cos 3x \, dx &= \frac{1}{2} \left[\int \sin 5x \, dx - \int \sin x \, dx \right] \\ &= \frac{1}{2} \left[-\frac{1}{5} \cos 5x + \cos x \right] + C \\ &= -\frac{1}{10} \cos 5x + \frac{1}{2} \cos x + C \end{aligned}$$

- (iii) From the identity
- $\sin 3x = 3 \sin x - 4 \sin^3 x$
- , we find that

$$\sin^3 x = \frac{3 \sin x - \sin 3x}{4}$$

$$\begin{aligned} \text{Therefore, } \int \sin^3 x \, dx &= \frac{3}{4} \int \sin x \, dx - \frac{1}{4} \int \sin 3x \, dx \\ &= -\frac{3}{4} \cos x + \frac{1}{12} \cos 3x + C \end{aligned}$$

$$\text{Alternatively, } \int \sin^3 x \, dx = \int \sin^2 x \sin x \, dx = \int (1 - \cos^2 x) \sin x \, dx$$

Put $\cos x = t$ so that $-\sin x \, dx = dt$

$$\begin{aligned} \text{Therefore, } \int \sin^3 x \, dx &= -\int (1 - t^2) \, dt = -\int dt + \int t^2 \, dt = -t + \frac{t^3}{3} + C \\ &= -\cos x + \frac{1}{3} \cos^3 x + C \end{aligned}$$

Remark It can be shown using trigonometric identities that both answers are equivalent.

EXERCISE 7.3

Find the integrals of the functions in Exercises 1 to 22:

- | | | |
|---|---|--|
| 1. $\sin^2(2x + 5)$ | 2. $\sin 3x \cos 4x$ | 3. $\cos 2x \cos 4x \cos 6x$ |
| 4. $\sin^3(2x + 1)$ | 5. $\sin^3 x \cos^3 x$ | 6. $\sin x \sin 2x \sin 3x$ |
| 7. $\sin 4x \sin 8x$ | 8. $\frac{1 - \cos x}{1 + \cos x}$ | 9. $\frac{\cos x}{1 + \cos x}$ |
| 10. $\sin^4 x$ | 11. $\cos^4 2x$ | 12. $\frac{\sin^2 x}{1 + \cos x}$ |
| 13. $\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$ | 14. $\frac{\cos x - \sin x}{1 + \sin 2x}$ | 15. $\tan^3 2x \sec 2x$ |
| 16. $\tan^4 x$ | 17. $\frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$ | 18. $\frac{\cos 2x + 2\sin^2 x}{\cos^2 x}$ |
| 19. $\frac{1}{\sin x \cos^3 x}$ | 20. $\frac{\cos 2x}{(\cos x + \sin x)^2}$ | 21. $\sin^{-1}(\cos x)$ |
| 22. $\frac{1}{\cos(x-a)\cos(x-b)}$ | | |

Choose the correct answer in Exercises 23 and 24.

23. $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx$ is equal to
- | | |
|----------------------------|---|
| (A) $\tan x + \cot x + C$ | (B) $\tan x + \operatorname{cosec} x + C$ |
| (C) $-\tan x + \cot x + C$ | (D) $\tan x + \sec x + C$ |
24. $\int \frac{e^x(1+x)}{\cos^2(e^x x)} dx$ equals
- | | |
|--------------------------|----------------------|
| (A) $-\cot(e^{x^x}) + C$ | (B) $\tan(xe^x) + C$ |
| (C) $\tan(e^x) + C$ | (D) $\cot(e^x) + C$ |

7.4 Integrals of Some Particular Functions

In this section, we mention below some important formulae of integrals and apply them for integrating many other related standard integrals:

$$(1) \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$$