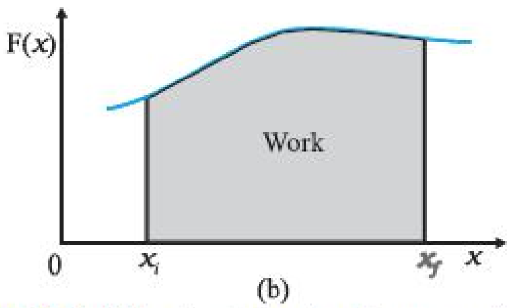

Fig. 6.3 *(a) The shaded rectangle represents the work done by the varying force F(x), over the small displacement Δx, ΔW = F(x) Δx. (b) adding the areas of all the rectangles we find that for Δx → 0, the area under the curve is exactly equal to the work done by F(x).*

▶ ***Example 6.5*** A woman pushes a trunk on a railway platform which has a rough surface. She applies a force of 100 N over a distance of 10 m. Thereafter, she gets progressively tired and her applied force reduces linearly with distance to 50 N. The total distance through which the trunk has been moved is 20 m. Plot the force applied by the woman and the frictional force, which is 50 N versus displacement. Calculate the work done by the two forces over 20 m.

Answer

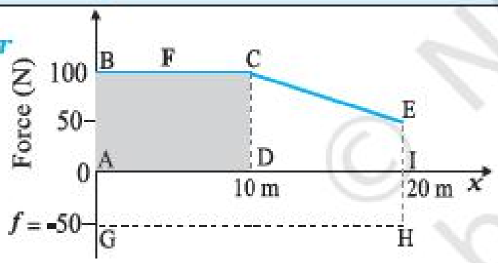

Fig. 6.4 *Plot of the force F applied by the woman and the opposing frictional force f versus displacement.*

The plot of the applied force is shown in Fig. 6.4. At x = 20 m, F = 50 N ($\neq 0$). We are given that the frictional force f is $|\mathbf{f}| = 50$ N. It opposes motion and acts in a direction opposite to **F**. It is therefore, shown on the negative side of the force axis.

The work done by the woman is

$W_F \rightarrow$ area of the rectangle ABCD + area of the trapezium CEID

$$W_F = 100 \times 10 + \frac{1}{2}(100 + 50) \times 10$$
$$= 1000 + 750$$
$$= 1750 \text{ J}$$

The work done by the frictional force is

$W_f \rightarrow$ area of the rectangle AGHI

$$W_f = (-50) \times 20$$
$$= -1000 \text{ J}$$

The area on the negative side of the force axis has a negative sign. ◀

6.6 THE WORK-ENERGY THEOREM FOR A VARIABLE FORCE

We are now familiar with the concepts of work and kinetic energy to prove the work-energy theorem for a variable force. We confine ourselves to one dimension. The time rate of change of kinetic energy is

$$\frac{dK}{dt} = \frac{d}{dt}\left(\frac{1}{2} m v^2\right)$$
$$= m \frac{dv}{dt} v$$
$$= F v \text{ (from Newton's Second Law)}$$
$$= F \frac{dx}{dt}$$

Thus

$$dK = F dx$$

Integrating from the initial position (x_i) to final position (x_f), we have

$$\int_{K_i}^{K_f} dK = \int_{x_i}^{x_f} F dx$$

where, K_i and K_f are the initial and final kinetic energies corresponding to x_i and x_f.

or $$K_f - K_i = \int_{x_i}^{x_f} F dx \quad (6.8a)$$

From Eq. (6.7), it follows that

$$K_f - K_i = W \quad (6.8b)$$

Thus, the WE theorem is proved for a variable force.

While the WE theorem is useful in a variety of problems, it does not, in general, incorporate the complete dynamical information of Newton's second law. It is an integral form of Newton's second law. Newton's second law is a relation between acceleration and force at any instant of time. Work-energy theorem involves an integral over an interval of time. In this sense, the temporal (time) information contained in the statement of Newton's second law is 'integrated over' and is not available explicitly. Another observation is that Newton's second law for two or three dimensions is in vector form whereas the work-energy theorem is in scalar form. In the scalar form, information with respect to directions contained in Newton's second law is not present.

Example 6.6 A block of mass $m = 1$ kg, moving on a horizontal surface with speed $v_i = 2$ ms^{-1} enters a rough patch ranging from $x = 0.10$ m to $x = 2.01$ m. The retarding force F_r on the block in this range is inversely proportional to x over this range,

$F_r = \dfrac{-k}{x}$ for $0.1 < x < 2.01$ m

$= 0$ for $x < 0.1$m and $x > 2.01$ m

where $k = 0.5$ J. What is the final kinetic energy and speed v_f of the block as it crosses this patch ?

Answer From Eq. (6.8a)

$$K_f = K_i + \int_{0.1}^{2.01} \frac{(-k)}{x} \mathrm{d}x$$

$$= \frac{1}{2} m v_i^2 - k \ln(x)\Big|_{0.1}^{2.01}$$

$$= \frac{1}{2} m v_i^2 - k \ln(2.01/0.1)$$

$$= 2 - 0.5 \ln(20.1)$$

$$= 2 - 1.5 = 0.5 \text{ J}$$

$$v_f = \sqrt{2K_f/m} = 1 \text{ m s}^{-1}$$

Here, note that ln is a symbol for the natural logarithm to the base e and not the logarithm to the base 10 [ln X = $\log_e$ X = 2.303 $\log_{10}$ X]. ◀

6.7 THE CONCEPT OF POTENTIAL ENERGY

The word potential suggests possibility or capacity for action. The term potential energy brings to one's mind 'stored' energy. A stretched bow-string possesses potential energy. When it is released, the arrow flies off at a great speed. The earth's crust is not uniform, but has discontinuities and dislocations that are called fault lines. These fault lines in the earth's crust are like 'compressed springs'. They possess a large amount of potential energy. An earthquake results when these fault lines readjust. Thus, potential energy is the 'stored energy' by virtue of the position or configuration of a body. The body left to itself releases this stored energy in the form of kinetic energy. Let us make our notion of potential energy more concrete.

The gravitational force on a ball of mass m is mg . g may be treated as a constant near the earth surface. By 'near' we imply that the height h of the ball above the earth's surface is very small compared to the earth's radius R_E ($h << R_E$) so that we can ignore the variation of g near the earth's surface*. In what follows we have taken the upward direction to be positive. Let us raise the ball up to a height h. The work done by the external agency against the gravitational force is mgh. This work gets stored as potential energy. Gravitational potential energy of an object, as a function of the height h, is denoted by $V(h)$ and it is the negative of work done by the gravitational force in raising the object to that height.

$V(h) = mgh$

If h is taken as a variable, it is easily seen that the gravitational force F equals the negative of the derivative of $V(h)$ with respect to h. Thus,

$$F = -\frac{\mathrm{d}}{\mathrm{d}h} V(h) = -m\,g$$

The negative sign indicates that the gravitational force is downward. When released, the ball comes down with an increasing speed. Just before it hits the ground, its speed is given by the kinematic relation,

$v^2 = 2gh$

This equation can be written as

$$\frac{1}{2} m\, v^2 = m\, g\, h$$

which shows that the gravitational potential energy of the object at height h, when the object is released, manifests itself as kinetic energy of the object on reaching the ground.

Physically, the notion of potential energy is applicable only to the class of forces where work done against the force gets 'stored up' as energy. When external constraints are removed, it manifests itself as kinetic energy. Mathematically, (for simplicity, in one dimension) the potential

* *The variation of g with height is discussed in Chapter 8 on Gravitation.*