The non-negative number

$$\sigma_x = \sqrt{\text{Var}(X)} = \sqrt{\sum_{i=1}^{n} (x_i - \mu)^2\, p(x_i)}$$

is called the *standard deviation* of the random variable X.

Another formula to find the variance of a random variable. We know that,

$$\text{Var}(X) = \sum_{i=1}^{n} (x_i - \mu)^2\, p(x_i)$$

$$= \sum_{i=1}^{n} (x_i^2 + \mu^2 - 2\mu x_i)\, p(x_i)$$

$$= \sum_{i=1}^{n} x_i^2\, p(x_i) + \sum_{i=1}^{n} \mu^2\, p(x_i) - \sum_{i=1}^{n} 2\mu x_i\, p(x_i)$$

$$= \sum_{i=1}^{n} x_i^2\, p(x_i) + \mu^2 \sum_{i=1}^{n} p(x_i) - 2\mu \sum_{i=1}^{n} x_i\, p(x_i)$$

$$= \sum_{i=1}^{n} x_i^2\, p(x_i) + \mu^2 - 2\mu^2 \left[\text{since} \sum_{i=1}^{n} p\,(x_i) = 1 \text{ and } \mu = \sum_{i=1}^{n} x_i\, p(x_i)\right]$$

$$= \sum_{i=1}^{n} x_i^2\, p(x_i) - \mu^2$$

or
$$\text{Var}(X) = \sum_{i=1}^{n} x_i^2\, p(x_i) - \left(\sum_{i=1}^{n} x_i\, p(x_i)\right)^2$$

or
$$\text{Var}(X) = E(X^2) - [E(X)]^2, \text{ where } E(X^2) = \sum_{i=1}^{n} x_i^2\, p(x_i)$$

Example 28 Find the variance of the number obtained on a throw of an unbiased die.

Solution The sample space of the experiment is S = {1, 2, 3, 4, 5, 6}.

Let X denote the number obtained on the throw. Then X is a random variable which can take values 1, 2, 3, 4, 5, or 6.

Also $$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$$

Therefore, the Probability distribution of X is

X	1	2	3	4	5	6
P(X)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Now $$E(X) = \sum_{i=1}^{n} x_i\, p(x_i)$$

$$= 1\times\frac{1}{6}+2\times\frac{1}{6}+3\times\frac{1}{6}+4\times\frac{1}{6}+5\times\frac{1}{6}+6\times\frac{1}{6}=\frac{21}{6}$$

Also $$E(X^2) = 1^2\times\frac{1}{6}+2^2\times\frac{1}{6}+3^2\times\frac{1}{6}+4^2\times\frac{1}{6}+5^2\times\frac{1}{6}+6^2\times\frac{1}{6}=\frac{91}{6}$$

Thus, $$\text{Var}(X) = E(X^2) - (E(X))^2$$

$$= \frac{91}{6}-\left(\frac{21}{6}\right)^2=\frac{91}{6}-\frac{441}{36} = \frac{35}{12}$$

Example 29 Two cards are drawn simultaneously (or successively without replacement) from a well shuffled pack of 52 cards. Find the mean, variance and standard deviation of the number of kings.

Solution Let X denote the number of kings in a draw of two cards. X is a random variable which can assume the values 0, 1 or 2.

Now $$P(X = 0) = P\text{ (no king)} = \frac{^{48}C_2}{^{52}C_2} = \frac{\frac{48!}{2!(48-2)!}}{\frac{52!}{2!(52-2)!}} = \frac{48\times 47}{52\times 51} = \frac{188}{221}$$

$$P(X = 1) = P\text{ (one king and one non-king)} = \frac{^{4}C_1\ ^{48}C_1}{^{52}C_2}$$

$$= \frac{4\times 48\times 2}{52\times 51} = \frac{32}{221}$$

and $$P(X = 2) = P(\text{two kings}) = \frac{^4C_2}{^{52}C_2} = \frac{4\times3}{52\times51} = \frac{1}{221}$$

Thus, the probability distribution of X is

X	0	1	2
P(X)	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

Now $$\text{Mean of } X = E(X) = \sum_{i=1}^{n} x_i\, p(x_i)$$

$$= 0\times\frac{188}{221} + 1\times\frac{32}{221} + 2\times\frac{1}{221} = \frac{34}{221}$$

Also $$E(X^2) = \sum_{i=1}^{n} x_i^2\, p(x_i)$$

$$= 0^2\times\frac{188}{221} + 1^2\times\frac{32}{221} + 2^2\times\frac{1}{221} = \frac{36}{221}$$

Now $$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= \frac{36}{221} - \left(\frac{34}{221}\right)^2 = \frac{6800}{(221)^2}$$

Therefore $$\sigma_x = \sqrt{\text{Var}(X)} = \frac{\sqrt{6800}}{221} = 0.37$$

EXERCISE 13.4

1. State which of the following are not the probability distributions of a random variable. Give reasons for your answer.

(i)

X	0	1	2
P(X)	0.4	0.4	0.2

(ii)

X	0	1	2	3	4
P(X)	0.1	0.5	0.2	– 0.1	0.3

13.7 Bernoulli Trials and Binomial Distribution

13.7.1 *Bernoulli trials*

Many experiments are dichotomous in nature. For example, a tossed coin shows a 'head' or 'tail', a manufactured item can be 'defective' or 'non-defective', the response to a question might be 'yes' or 'no', an egg has 'hatched' or 'not hatched', the decision is 'yes' or 'no' etc. In such cases, it is customary to call one of the outcomes a 'success' and the other 'not success' or 'failure'. For example, in tossing a coin, if the occurrence of the head is considered a success, then occurrence of tail is a failure.

Each time we toss a coin or roll a die or perform any other experiment, we call it a trial. If a coin is tossed, say, 4 times, the number of trials is 4, each having exactly two outcomes, namely, success or failure. The outcome of any trial is independent of the outcome of any other trial. In each of such trials, the probability of success or failure remains constant. Such independent trials which have only two outcomes usually referred as 'success' or 'failure' are called *Bernoulli trials.*

Definition 8 Trials of a random experiment are called Bernoulli trials, if they satisfy the following conditions :

(i) There should be a finite number of trials.

(ii) The trials should be independent.

(iii) Each trial has exactly two outcomes : success or failure.

(iv) The probability of success remains the same in each trial.

For example, throwing a die 50 times is a case of 50 Bernoulli trials, in which each trial results in success (say an even number) or failure (an odd number) and the probability of success (p) is same for all 50 throws. Obviously, the successive throws of the die are independent experiments. If the die is fair and have six numbers 1 to 6 written on six faces, then $p = \frac{1}{2}$ and $q = 1 - p = \frac{1}{2}$ = probability of failure.

Example 30 Six balls are drawn successively from an urn containing 7 red and 9 black balls. Tell whether or not the trials of drawing balls are Bernoulli trials when after each draw the ball drawn is

(i) replaced (ii) not replaced in the urn.

Solution

(i) The number of trials is finite. When the drawing is done with replacement, the probability of success (say, red ball) is $p = \frac{7}{16}$ which is same for all six trials (draws). Hence, the drawing of balls with replacements are Bernoulli trials.

(ii) When the drawing is done without replacement, the probability of success (i.e., red ball) in first trial is $\frac{7}{16}$, in 2nd trial is $\frac{6}{15}$ if the first ball drawn is red or $\frac{7}{15}$ if the first ball drawn is black and so on. Clearly, the probability of success is not same for all trials, hence the trials are not Bernoulli trials.

13.7.2 *Binomial distribution*

Consider the experiment of tossing a coin in which each trial results in success (say, heads) or failure (tails). Let S and F denote respectively success and failure in each trial. Suppose we are interested in finding the ways in which we have one success in six trials.

Clearly, six different cases are there as listed below:

SFFFFF, FSFFFF, FFSFFF, FFFSFF, FFFFSF, FFFFFS.

Similarly, two successes and four failures can have $\frac{6!}{4! \times 2!}$ combinations. It will be lengthy job to list all of these ways. Therefore, calculation of probabilities of 0, 1, 2,..., n number of successes may be lengthy and time consuming. To avoid the lengthy calculations and listing of all the possible cases, for the probabilities of number of successes in n-Bernoulli trials, a formula is derived. For this purpose, let us take the experiment made up of three Bernoulli trials with probabilities p and $q = 1 - p$ for success and failure respectively in each trial. The sample space of the experiment is the set

$$S = \{SSS, SSF, SFS, FSS, SFF, FSF, FFS, FFF\}$$

The number of successes is a random variable X and can take values 0, 1, 2, or 3. The probability distribution of the number of successes is as below :

$$\begin{aligned}
P(X = 0) &= P(\text{no success}) \\
&= P(\{FFF\}) = P(F)\ P(F)\ P(F) \\
&= q \cdot q \cdot q = q^3 \text{ since the trials are independent} \\
P(X = 1) &= P(\text{one successes}) \\
&= P(\{SFF, FSF, FFS\}) \\
&= P(\{SFF\}) + P(\{FSF\}) + P(\{FFS\}) \\
&= P(S)\ P(F)\ P(F) + P(F)\ P(S)\ P(F) + P(F)\ P(F)\ P(S) \\
&= p.q.q + q.p.q + q.q.p = 3pq^2 \\
P(X = 2) &= P\ (\text{two successes}) \\
&= P(\{SSF, SFS, FSS\}) \\
&= P(\{SSF\}) + P\ (\{SFS\}) + P(\{FSS\})
\end{aligned}$$

$= P(S)\ P(S)\ P(F) + P(S)\ P(F)\ P(S) + P(F)\ P(S)\ P(S)$

$= p.p.q. + p.q.p + q.p.p = 3p^2q$

and $P(X = 3) = P(\text{three success}) = P(\{SSS\})$

$= P(S)\ .\ P(S)\ .\ P(S) = p^3$

Thus, the probability distribution of X is

X	0	1	2	3
P(X)	q^3	$3q^2p$	$3qp^2$	p^3

Also, the binominal expansion of $(q + p)^3$ is

$$q^3 + 3q^2p + 3qp^2 + p^3$$

Note that the probabilities of 0, 1, 2 or 3 successes are respectively the 1st, 2nd, 3rd and 4th term in the expansion of $(q + p)^3$.

Also, since $q + p = 1$, it follows that the sum of these probabilities, as expected, is 1.

Thus, we may conclude that in an experiment of n-Bernoulli trials, the probabilities of 0, 1, 2,..., n successes can be obtained as 1st, 2nd,...,$(n + 1)^{th}$ terms in the expansion of $(q + p)^n$. To prove this assertion (result), let us find the probability of x-successes in an experiment of n-Bernoulli trials.

Clearly, in case of x successes (S), there will be $(n - x)$ failures (F).

Now, x successes (S) and $(n - x)$ failures (F) can be obtained in $\dfrac{n!}{x!(n-x)!}$ ways.

In each of these ways, the probability of x successes and $(n - x)$ failures is

$= P(x \text{ successes})\ .\ P(n-x) \text{ failures is}$

$$= \underbrace{P(S).P(S)...P(S)}_{x\,\text{times}} \cdot \underbrace{P(F).P(F)...P(F)}_{(n-x)\,\text{times}} = p^x\, q^{n-x}$$

Thus, the probability of x successes in n-Bernoulli trials is $\dfrac{n!}{x!(n-x)!} p^x\, q^{n-x}$

or ${}^nC_x\, p^x\, q^{n-x}$

Thus $P(x \text{ successes}) = {}^nC_x\, p^x\, q^{n-x}, \quad x = 0, 1, 2,...,n.\ (q = 1 - p)$

Clearly, $P(x \text{ successes})$, i.e. ${}^nC_x\, p^x\, q^{n-x}$ is the $(x + 1)^{th}$ term in the binomial expansion of $(q + p)^n$.

Thus, the probability distribution of number of successes in an experiment consisting of n Bernoulli trials may be obtained by the binomial expansion of $(q + p)^n$. Hence, this distribution of number of successes X can be written as

X	0	1	2	...	x	...	n
P(X)	${}^nC_0\, q^n$	${}^nC_1\, q^{n-1}p^1$	${}^nC_2\, q^{n-2}p^2$		${}^nC_x\, q^{n-x}p^x$		${}^nC_n\, p^n$

The above probability distribution is known as *binomial distribution* with parameters n and p, because for given values of n and p, we can find the complete probability distribution.

The probability of x successes $P(X = x)$ is also denoted by $P(x)$ and is given by

$$P(x) = {}^nC_x q^{n-x}p^x, \quad x = 0, 1, ..., n. \ (q = 1 - p)$$

This $P(x)$ is called the *probability function* of the binomial distribution.

A binomial distribution with n-Bernoulli trials and probability of success in each trial as p, is denoted by $B(n, p)$.

Let us now take up some examples.

Example 31 If a fair coin is tossed 10 times, find the probability of

(i) exactly six heads

(ii) at least six heads

(iii) at most six heads

Solution The repeated tosses of a coin are Bernoulli trials. Let X denote the number of heads in an experiment of 10 trials.

Clearly, X has the binomial distribution with $n = 10$ and $p = \frac{1}{2}$

Therefore $$P(X = x) = {}^nC_x q^{n-x}p^x, \ x = 0, 1, 2, ..., n$$

Here $$n = 10, \ p = \frac{1}{2}, \ q = 1 - p = \frac{1}{2}$$

Therefore $$P(X = x) = {}^{10}C_x \left(\frac{1}{2}\right)^{10-x} \left(\frac{1}{2}\right)^x = {}^{10}C_x \left(\frac{1}{2}\right)^{10}$$

Now (i) $$P(X = 6) = {}^{10}C_6 \left(\frac{1}{2}\right)^{10} = \frac{10!}{6! \times 4!} \frac{1}{2^{10}} = \frac{105}{512}$$

(ii) $P(\text{at least six heads}) = P(X \geq 6)$

$$= P(X = 6) + P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)$$

$$= {}^{10}C_6\left(\frac{1}{2}\right)^{10} + {}^{10}C_7\left(\frac{1}{2}\right)^{10} + {}^{10}C_8\left(\frac{1}{2}\right)^{10} + {}^{10}C_9\left(\frac{1}{2}\right)^{10} + {}^{10}C_{10}\left(\frac{1}{2}\right)^{10}$$

$$= \left[\left(\frac{10!}{6!\times 4!}\right) + \left(\frac{10!}{7!\times 3!}\right) + \left(\frac{10!}{8!\times 2!}\right) + \left(\frac{10!}{9!\times 1!}\right) + \left(\frac{10!}{10!}\right)\right]\frac{1}{2^{10}} = \frac{193}{512}$$

(iii) P(at most six heads) = $P(X \le 6)$

$$= P(X=0) + P(X=1) + P(X=2) + P(X=3)$$
$$+ P(X=4) + P(X=5) + P(X=6)$$

$$= \left(\frac{1}{2}\right)^{10} + {}^{10}C_1\left(\frac{1}{2}\right)^{10} + {}^{10}C_2\left(\frac{1}{2}\right)^{10} + {}^{10}C_3\left(\frac{1}{2}\right)^{10}$$
$$+ {}^{10}C_4\left(\frac{1}{2}\right)^{10} + {}^{10}C_5\left(\frac{1}{2}\right)^{10} + {}^{10}C_6\left(\frac{1}{2}\right)^{10}$$

$$= \frac{848}{1024} = \frac{53}{64}$$

Example 32 Ten eggs are drawn successively with replacement from a lot containing 10% defective eggs. Find the probability that there is at least one defective egg.

Solution Let X denote the number of defective eggs in the 10 eggs drawn. Since the drawing is done with replacement, the trials are Bernoulli trials. Clearly, X has the binomial distribution with $n = 10$ and $p = \frac{10}{100} = \frac{1}{10}$.

Therefore $q = 1 - p = \frac{9}{10}$

Now P(at least one defective egg) = $P(X \ge 1) = 1 - P(X = 0)$

$$= 1 - {}^{10}C_0\left(\frac{9}{10}\right)^{10} = 1 - \frac{9^{10}}{10^{10}}$$

EXERCISE 13.5

1. A die is thrown 6 times. If 'getting an odd number' is a success, what is the probability of
   (i) 5 successes? (ii) at least 5 successes?
   (iii) at most 5 successes?