We observe that the maximum profit to the dealer results from the investment strategy (10, 50), i.e. buying 10 tables and 50 chairs.

This method of solving linear programming problem is referred as **Corner Point Method**. The method comprises of the following steps:

1. Find the feasible region of the linear programming problem and determine its corner points (vertices) either by inspection or by solving the two equations of the lines intersecting at that point.
2. Evaluate the objective function $Z = ax + by$ at each corner point. Let M and m, respectively denote the largest and smallest values of these points.
3. (i) When the feasible region is **bounded**, M and m are the maximum and minimum values of Z.

   (ii) In case, the feasible region is **unbounded**, we have:
4. (a) M is the maximum value of Z, if the open half plane determined by $ax + by > M$ has no point in common with the feasible region. Otherwise, Z has no maximum value.

   (b) Similarly, m is the minimum value of Z, if the open half plane determined by $ax + by < m$ has no point in common with the feasible region. Otherwise, Z has no minimum value.

We will now illustrate these steps of Corner Point Method by considering some examples:

Example 1 Solve the following linear programming problem graphically:

Maximise $Z = 4x + y$ ... (1)

subject to the constraints:

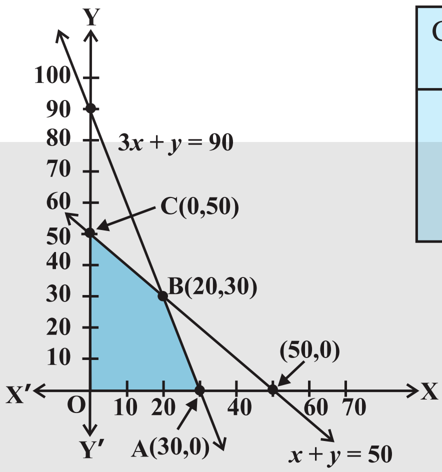
$$x + y \leq 50 \quad \text{... (2)}$$

$$3x + y \leq 90 \quad \text{... (3)}$$

$$x \geq 0, y \geq 0 \quad \text{... (4)}$$

Solution The shaded region in Fig 12.2 is the feasible region determined by the system of constraints (2) to (4). We observe that the feasible region OABC is **bounded.** So, we now use Corner Point Method to determine the maximum value of Z.

The coordinates of the corner points O, A, B and C are (0, 0), (30, 0), (20, 30) and (0, 50) respectively. Now we evaluate Z at each corner point.



Corner Point	Corresponding value of Z	
(0, 0)	0	
(30, 0)	**120** ←	Maximum
(20, 30)	110	
(0, 50)	50	

Fig 12.2

Hence, maximum value of Z is 120 at the point (30, 0).

Example 2 Solve the following linear programming problem graphically:

Minimise Z = 200 x + 500 y ... (1)

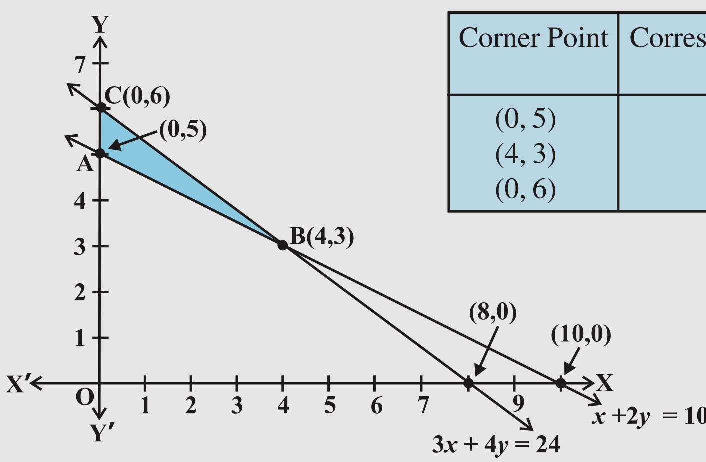
subject to the constraints:

$$x + 2y \geq 10 \quad \text{... (2)}$$

$$3x + 4y \leq 24 \quad \text{... (3)}$$

$$x \geq 0,\ y \geq 0 \quad \text{... (4)}$$

Solution The shaded region in Fig 12.3 is the feasible region ABC determined by the system of constraints (2) to (4), which is **bounded**. The coordinates of corner points



Corner Point	Corresponding value of Z	
(0, 5)	2500	
(4, 3)	**2300** ←	Minimum
(0, 6)	3000	

Fig 12.3

A, B and C are (0,5), (4,3) and (0,6) respectively. Now we evaluate $Z = 200x + 500y$ at these points.

Hence, minimum value of Z is 2300 attained at the point (4, 3)

Example 3 Solve the following problem graphically:

Minimise and Maximise $Z = 3x + 9y$ ... (1)

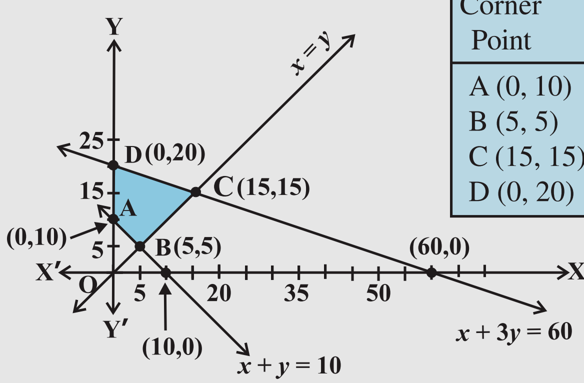
subject to the constraints: $x + 3y \leq 60$ ... (2)

$x + y \geq 10$ ... (3)

$x \leq y$ ... (4)

$x \geq 0, y \geq 0$ ... (5)

Solution First of all, let us graph the feasible region of the system of linear inequalities (2) to (5). The feasible region ABCD is shown in the Fig 12.4. Note that the region is bounded. The coordinates of the corner points A, B, C and D are (0, 10), (5, 5), (15,15) and (0, 20) respectively.



Corner Point	Corresponding value of $Z = 3x + 9y$	
A (0, 10)	90	
B (5, 5)	**60** ←	Minimum
C (15, 15)	**180** } ←	Maximum (Multiple optimal solutions)
D (0, 20)	**180**	

Fig 12.4

We now find the minimum and maximum value of Z. From the table, we find that the minimum value of Z is 60 at the point B (5, 5) of the feasible region.

The maximum value of Z on the feasible region occurs at the two corner points C (15, 15) and D (0, 20) and it is 180 in each case.

Remark Observe that in the above example, the problem has multiple optimal solutions at the corner points C and D, i.e. the both points produce same maximum value 180. In such cases, you can see that every point on the line segment CD joining the two corner points C and D also give the same maximum value. Same is also true in the case if the two points produce same minimum value.

Example 4 Determine graphically the minimum value of the objective function

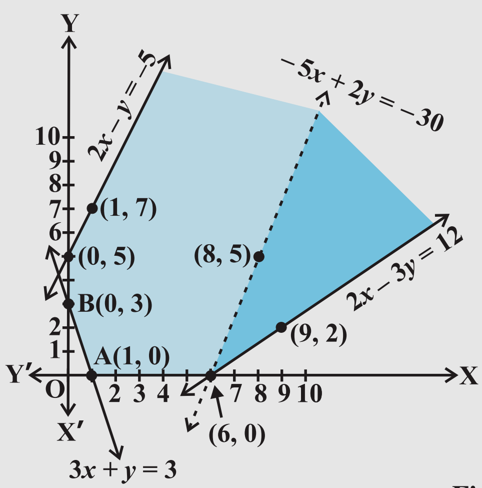
$$Z = -50x + 20y \quad \text{... (1)}$$

subject to the constraints:

$$2x - y \geq -5 \quad \text{... (2)}$$
$$3x + y \geq 3 \quad \text{... (3)}$$
$$2x - 3y \leq 12 \quad \text{... (4)}$$
$$x \geq 0, y \geq 0 \quad \text{... (5)}$$

Solution First of all, let us graph the feasible region of the system of inequalities (2) to (5). The feasible region (shaded) is shown in the Fig 12.5. Observe that the feasible region is **unbounded**.

We now evaluate Z at the corner points.



Corner Point	$Z = -50x + 20y$
(0, 5)	100
(0, 3)	**60**
(1, 0)	**–50**
(6, 0)	**– 300** ⟵ smallest

Fig 12.5

From this table, we find that – 300 is the smallest value of Z at the corner point (6, 0). Can we say that minimum value of Z is – 300? Note that if the region would have been bounded, this smallest value of Z is the minimum value of Z (Theorem 2). But here we see that the feasible region is unbounded. Therefore, – 300 may or may not be the minimum value of Z. To decide this issue, we graph the inequality

$$-50x + 20y < -300 \text{ (see Step 3(ii) of corner Point Method.)}$$

i.e.,
$$-5x + 2y < -30$$

and check whether the resulting open half plane has points in common with feasible region or not. If it has common points, then –300 will not be the minimum value of *Z*. Otherwise, –300 will be the minimum value of Z.

As shown in the Fig 12.5, it has common points. Therefore, Z = –50 x + 20 y has no minimum value subject to the given constraints.

In the above example, can you say whether $z = -50x + 20y$ has the maximum value 100 at (0,5)? For this, check whether the graph of $-50x + 20y > 100$ has points in common with the feasible region. (Why?)

Example 5 Minimise Z = $3x + 2y$

subject to the constraints:

$$x + y \geq 8 \quad \text{... (1)}$$

$$3x + 5y \leq 15 \quad \text{... (2)}$$

$$x \geq 0, y \geq 0 \quad \text{... (3)}$$

Solution Let us graph the inequalities (1) to (3) (Fig 12.6). Is there any feasible region? Why is so?

From Fig 12.6, you can see that there is no point satisfying all the constraints simultaneously. Thus, the problem is having no feasible region and hence no feasible solution.

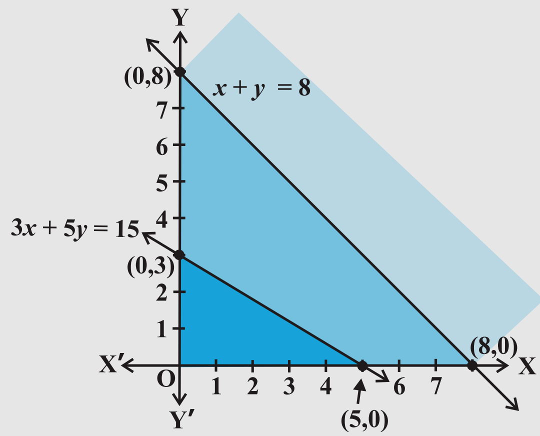


Fig 12.6

Remarks From the examples which we have discussed so far, we notice some general features of linear programming problems:

(i) The feasible region is always a convex region.

(ii) The maximum (or minimum) solution of the objective function occurs at the vertex (corner) of the feasible region. If two corner points produce the same maximum (or minimum) value of the objective function, then every point on the line segment joining these points will also give the same maximum (or minimum) value.

EXERCISE 12.1

Solve the following Linear Programming Problems graphically:

1. Maximise Z = $3x + 4y$

subject to the constraints : $x + y \leq 4$, $x \geq 0$, $y \geq 0$.