

$$\text{Hence} \quad \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

Similarly, we can prove the other parts.

$$5. \text{ (i) } \tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}, \quad xy < 1$$

$$\text{(ii) } \tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy}, \quad xy > -1$$

$$\text{(iii) } \tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1} \left(\frac{x+y}{1-xy} \right), \quad xy > 1; \quad x, y > 0$$

Let $\tan^{-1} x = \theta$ and $\tan^{-1} y = \phi$. Then $x = \tan \theta$, $y = \tan \phi$

$$\text{Now} \quad \tan(\theta + \phi) = \frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi} = \frac{x+y}{1-xy}$$

$$\text{This gives} \quad \theta + \phi = \tan^{-1} \frac{x+y}{1-xy}$$

$$\text{Hence} \quad \tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$$

In the above result, if we replace y by $-y$, we get the second result and by replacing y by x , we get the third result as given below.

$$6. \text{ (i) } 2 \tan^{-1} x = \sin^{-1} \frac{2x}{1+x^2}, \quad |x| \leq 1$$

$$\text{(ii) } 2 \tan^{-1} x = \cos^{-1} \frac{1-x^2}{1+x^2}, \quad x \geq 0$$

$$\text{(iii) } 2 \tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2}, \quad -1 < x < 1$$

Let $\tan^{-1} x = y$, then $x = \tan y$. Now

$$\begin{aligned} \sin^{-1} \frac{2x}{1+x^2} &= \sin^{-1} \frac{2 \tan y}{1 + \tan^2 y} \\ &= \sin^{-1} (\sin 2y) = 2y = 2 \tan^{-1} x \end{aligned}$$

$$\text{Also } \cos^{-1} \frac{1-x^2}{1+x^2} = \cos^{-1} \frac{1-\tan^2 y}{1+\tan^2 y} = \cos^{-1} (\cos 2y) = 2y = 2\tan^{-1} x$$

(iii) Can be worked out similarly.

We now consider some examples.

Example 3 Show that

$$(i) \sin^{-1} (2x\sqrt{1-x^2}) = 2 \sin^{-1} x, \quad -\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$$

$$(ii) \sin^{-1} (2x\sqrt{1-x^2}) = 2 \cos^{-1} x, \quad \frac{1}{\sqrt{2}} \leq x \leq 1$$

Solution

(i) Let $x = \sin \theta$. Then $\sin^{-1} x = \theta$. We have

$$\begin{aligned} \sin^{-1} (2x\sqrt{1-x^2}) &= \sin^{-1} (2 \sin \theta \sqrt{1-\sin^2 \theta}) \\ &= \sin^{-1} (2 \sin \theta \cos \theta) = \sin^{-1} (\sin 2\theta) = 2\theta \\ &= 2 \sin^{-1} x \end{aligned}$$

(ii) Take $x = \cos \theta$, then proceeding as above, we get, $\sin^{-1} (2x\sqrt{1-x^2}) = 2 \cos^{-1} x$

Example 4 Show that $\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{2}{11} = \tan^{-1} \frac{3}{4}$

Solution By property 5 (i), we have

$$\text{L.H.S.} = \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{2}{11} = \tan^{-1} \frac{\frac{1}{2} + \frac{2}{11}}{1 - \frac{1}{2} \times \frac{2}{11}} = \tan^{-1} \frac{15}{20} = \tan^{-1} \frac{3}{4} = \text{R.H.S.}$$

Example 5 Express $\tan^{-1} \frac{\cos x}{1-\sin x}$, $-\frac{3\pi}{2} < x < \frac{\pi}{2}$ in the simplest form.

Solution We write

$$\tan^{-1} \left(\frac{\cos x}{1-\sin x} \right) = \tan^{-1} \left[\frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} - 2 \sin \frac{x}{2} \cos \frac{x}{2}} \right]$$

$$\begin{aligned}
 &= \tan^{-1} \left[\frac{\left(\cos \frac{x}{2} + \sin \frac{x}{2} \right) \left(\cos \frac{x}{2} - \sin \frac{x}{2} \right)}{\left(\cos \frac{x}{2} - \sin \frac{x}{2} \right)^2} \right] \\
 &= \tan^{-1} \left[\frac{\cos \frac{x}{2} + \sin \frac{x}{2}}{\cos \frac{x}{2} - \sin \frac{x}{2}} \right] = \tan^{-1} \left[\frac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}} \right] \\
 &= \tan^{-1} \left[\tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \right] = \frac{\pi}{4} + \frac{x}{2}
 \end{aligned}$$

Alternatively,

$$\begin{aligned}
 \tan^{-1} \left(\frac{\cos x}{1 - \sin x} \right) &= \tan^{-1} \left[\frac{\sin \left(\frac{\pi}{2} - x \right)}{1 - \cos \left(\frac{\pi}{2} - x \right)} \right] = \tan^{-1} \left[\frac{\sin \left(\frac{\pi - 2x}{2} \right)}{1 - \cos \left(\frac{\pi - 2x}{2} \right)} \right] \\
 &= \tan^{-1} \left[\frac{2 \sin \left(\frac{\pi - 2x}{4} \right) \cos \left(\frac{\pi - 2x}{4} \right)}{2 \sin^2 \left(\frac{\pi - 2x}{4} \right)} \right] \\
 &= \tan^{-1} \left[\cot \left(\frac{\pi - 2x}{4} \right) \right] = \tan^{-1} \left[\tan \left(\frac{\pi}{2} - \frac{\pi - 2x}{4} \right) \right] \\
 &= \tan^{-1} \left[\tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \right] = \frac{\pi}{4} + \frac{x}{2}
 \end{aligned}$$

Example 6 Write $\cot^{-1} \left(\frac{1}{\sqrt{x^2 - 1}} \right)$, $x > 1$ in the simplest form.

Solution Let $x = \sec \theta$, then $\sqrt{x^2 - 1} = \sqrt{\sec^2 \theta - 1} = \tan \theta$

Therefore, $\cot^{-1} \frac{1}{\sqrt{x^2-1}} = \cot^{-1}(\cot \theta) = \theta = \sec^{-1} x$, which is the simplest form.

Example 7 Prove that $\tan^{-1} x + \tan^{-1} \frac{2x}{1-x^2} = \tan^{-1} \left(\frac{3x-x^3}{1-3x^2} \right)$, $|x| < \frac{1}{\sqrt{3}}$

Solution Let $x = \tan \theta$. Then $\theta = \tan^{-1} x$. We have

$$\begin{aligned} \text{R.H.S.} &= \tan^{-1} \left(\frac{3x-x^3}{1-3x^2} \right) = \tan^{-1} \left(\frac{3 \tan \theta - \tan^3 \theta}{1-3 \tan^2 \theta} \right) \\ &= \tan^{-1} (\tan 3\theta) = 3\theta = 3 \tan^{-1} x = \tan^{-1} x + 2 \tan^{-1} x \\ &= \tan^{-1} x + \tan^{-1} \frac{2x}{1-x^2} = \text{L.H.S. (Why?)} \end{aligned}$$

Example 8 Find the value of $\cos(\sec^{-1} x + \operatorname{cosec}^{-1} x)$, $|x| \geq 1$

Solution We have $\cos(\sec^{-1} x + \operatorname{cosec}^{-1} x) = \cos\left(\frac{\pi}{2}\right) = 0$

EXERCISE 2.2

Prove the following:

1. $3 \sin^{-1} x = \sin^{-1} (3x - 4x^3)$, $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$
2. $3 \cos^{-1} x = \cos^{-1} (4x^3 - 3x)$, $x \in \left[\frac{1}{2}, 1\right]$
3. $\tan^{-1} \frac{2}{11} + \tan^{-1} \frac{7}{24} = \tan^{-1} \frac{1}{2}$
4. $2 \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{31}{17}$

Write the following functions in the simplest form:

5. $\tan^{-1} \frac{\sqrt{1+x^2}-1}{x}$, $x \neq 0$
6. $\tan^{-1} \frac{1}{\sqrt{x^2-1}}$, $|x| > 1$
7. $\tan^{-1} \left(\frac{\sqrt{1-\cos x}}{\sqrt{1+\cos x}} \right)$, $0 < x < \pi$
8. $\tan^{-1} \left(\frac{\cos x - \sin x}{\cos x + \sin x} \right)$, $-\frac{\pi}{4} < x < \frac{3\pi}{4}$

9. $\tan^{-1} \frac{x}{\sqrt{a^2 - x^2}}, |x| < a$
10. $\tan^{-1} \left(\frac{3a^2x - x^3}{a^3 - 3ax^2} \right), a > 0; \frac{-a}{\sqrt{3}} < x < \frac{a}{\sqrt{3}}$

Find the values of each of the following:

11. $\tan^{-1} \left[2 \cos \left(2 \sin^{-1} \frac{1}{2} \right) \right]$

12. $\cot (\tan^{-1}a + \cot^{-1}a)$

13. $\tan \frac{1}{2} \left[\sin^{-1} \frac{2x}{1+x^2} + \cos^{-1} \frac{1-y^2}{1+y^2} \right], |x| < 1, y > 0 \text{ and } xy < 1$

14. If $\sin \left(\sin^{-1} \frac{1}{5} + \cos^{-1} x \right) = 1$, then find the value of x

15. If $\tan^{-1} \frac{x-1}{x-2} + \tan^{-1} \frac{x+1}{x+2} = \frac{\pi}{4}$, then find the value of x

Find the values of each of the expressions in Exercises 16 to 18.

16. $\sin^{-1} \left(\sin \frac{2\pi}{3} \right)$

17. $\tan^{-1} \left(\tan \frac{3\pi}{4} \right)$

18. $\tan \left(\sin^{-1} \frac{3}{5} + \cot^{-1} \frac{3}{2} \right)$

19. $\cos^{-1} \left(\cos \frac{7\pi}{6} \right)$ is equal to

- (A) $\frac{7\pi}{6}$ (B) $\frac{5\pi}{6}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{6}$

20. $\sin \left(\frac{\pi}{3} - \sin^{-1} \left(-\frac{1}{2} \right) \right)$ is equal to

- (A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) 1

21. $\tan^{-1} \sqrt{3} - \cot^{-1} (-\sqrt{3})$ is equal to

- (A) π (B) $-\frac{\pi}{2}$ (C) 0 (D) $2\sqrt{3}$