Definite integral
lecture-4

Revision hill lecture

we studied $\int_a^b f(x)\,dx$
by 2 methods ① limit of sums
② Anti Derivatives.

there are integration which are very simple whose antiderivative exist and can be written in this form

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{where } \frac{d}{dx}F(x) = f'(x).$$

→ but there are several cases where $\int_a^b f(x)\,dx$ is very very complicated.
Difficult to compute antiderivative
i.e. $F'(x) = f(x)$ is difficult to find.
in this cases we applied method of substitution.

- we transform the integration in new form.

$$\int_a^b f(x)\,dx = \int_{a'}^{b'} g(t)\,dt$$

↳ where computing simple enough derivative.

- we also seem some 7 to 8 properties for definite integrals.
where computing integral very easy using properties. $\int_a^b f(x)\,dx$.

- we have computed many areas as application of definite integrals.

① circles. - Ellipses.

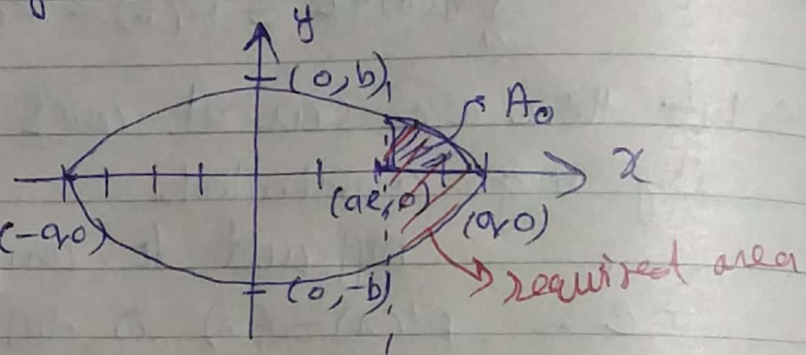
② Area of a curve bounded between the curve and a given line.

[05:20] more problems to understand Definite integrals

Ex: find out smaller area bounded between ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and vertical line $x=ae$
e - is eccentricity

where $a>b$.

Ans :- lets plot it first



area is symmetric about x-axis -
So required Area $= 2 \times A_0$ (Area of ▨ region)

So elemental Area is $2\int_{x=ae}^{x=a} y\,dx$

If we solve for y, we get $y=\pm b\sqrt{1-\frac{x^2}{a^2}}$

since $y>0$ we are using for area
use
$$y=b\sqrt{1-\frac{x^2}{a^2}} \Rightarrow \frac{b}{a}\sqrt{a^2-x^2}$$

so required area $= 2\int_{ae}^{a}\frac{b}{a}\sqrt{a^2-x^2}\,dx$

$= \frac{2b}{a}\int_{ae}^{a}\sqrt{a^2-x^2}\,dx = \frac{2b}{a}\left[\frac{1}{2}x\sqrt{a^2-x^2}+\frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right)\right]_{x=ae}^{x=a}$

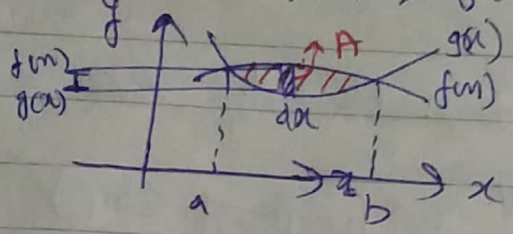
~~$= \frac{2b}{a}\left[0+\frac{1}{2}\sin^{-1}1-\frac{1}{2}ae\sqrt{a^2-a^2e^2}+\frac{1}{2}a^2\sin^{-1}e\right]$~~

$= \frac{2b}{a}\left[\frac{\pi a^2}{4}-\frac{1}{2}aeb-\frac{1}{2}a^2\sin^{-1}e\right]$

[12:42] another example:-

Area bounded between two curves.

case 1. $f(x) \geq g(x) \quad \forall x$ in $[a, b]$



so divide this in small Ractangle strips.

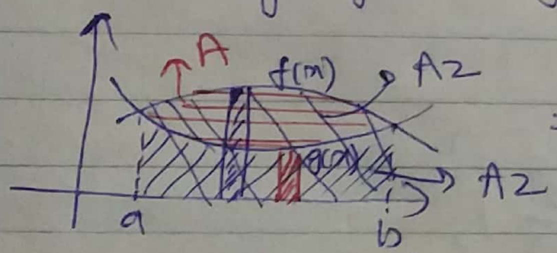
so area of elementary strip = dx(height of strip)

$dA = dx(f(x) - g(x))$

so total Area

$\Rightarrow A = \int_{x=a}^{b} [f(x) - g(x)]\,dx$

another way of looking to the same problem.



so Required Area = $A_2 - A_1$

$A_2 = \int_a^b f(x)\,dx$

which we know $\Rightarrow$

$A_1 = \int_a^b g(x)\,dx$

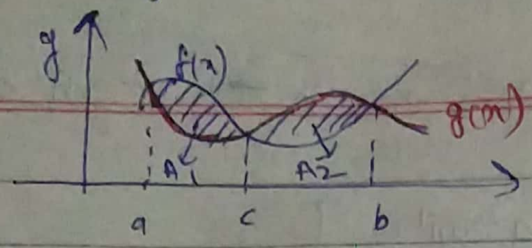
$\Rightarrow A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

case : 2 :

$f(x) \geq g(x) \quad \forall x \in [a, c]$

$g(x) \geq f(x) \quad x \in [c, b]$

graphically solution look like.

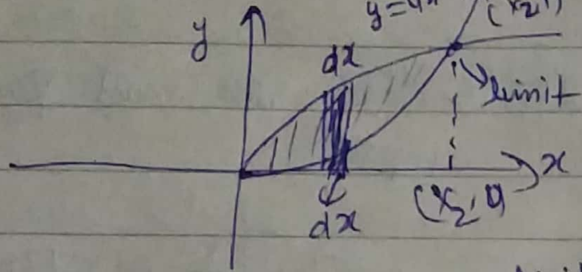


so required Area $= \underbrace{\int_a^c [f(x)-g(x)]\,dx}_{A_1} + \underbrace{\int_c^b [g(x)-f(x)]\,dx}_{A_2}$

[20:51]: Area example between the curves.

Example 1: Area bonded between $y^2 = 2x$ & $y = 4x^2$.

⇒ lets view the problem graphically



then elementry Area of this region is.

⇒ dx(hight of strip)

$dA \Rightarrow dx\,(\sqrt{2x} - 4x^2)\,dx$

so $A = \int dA = \int_0^{\text{limit}} (\sqrt{2x} - 4x^2)\,dx$

now limit is solution of point of intsection of

$y^2 = 2x$ and $y = 4x^2$

⇒ y^2 ⇒ $16x^4 = 2x$ ⇒ $x = 0$, $x = 1/2$.

so $A = \int_0^{1/2} (\sqrt{2x} - 4x^2)\,dx$

$= \sqrt{2}\, \frac{x^{3/2}}{3/2}\Big|_0^{1/2} - 4\frac{x^3}{3}\Big|_0^{1/2}$

$= \frac{2\sqrt{2}}{3} \frac{1}{2} \cdot \frac{1}{\sqrt{2}} - \frac{4}{3} \times \frac{1}{8}$

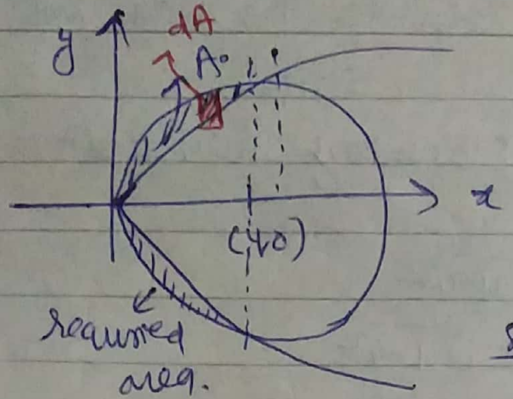
$= \frac{1}{3} - \frac{1}{6} \Rightarrow \frac{1}{6}$ ans.

[25:55] Another example.

Ex: find out area of region which is outside parabola $y^2 = 4x$ and inside the circle $x^2+y^2 = 8x$.

As the equation can be written as $(x-4)^2+y^2=16$.

lets plot the graph to look problem more easy way



and point of intersection is

$x^2+4x = 8x$

$\Rightarrow x^2 = 4x \Rightarrow x=0,\ x=4$

both circle and parabola are symmetric so area also symmetric.

So required area is $\Rightarrow$ $2 \cdot A$.

$dA = dx(\text{hight of strip})$

$= dx\left(\sqrt{16-(x-4)^2} - \sqrt{4x}\right)$

So total Area $\Rightarrow$ $A = \int dA = \int_0^{x=4} \left(\sqrt{16-(x-4)^2} - \sqrt{4x}\right) dx$.

$=$ substitute $x-4 = t$

$\Rightarrow \int_{-4}^{0} \left(\sqrt{16-t^2}\right) dt - 2\int_0^4 \sqrt{x}\, dx$

$= \frac{1}{2}\left[t\sqrt{16-t^2} + \frac{1}{2}\times 16 \sin^{-1}\left(\frac{t}{4}\right)\right]_{-4}^{4} - \left[2\frac{x^{3/2}}{3/2}\right]_0^4$

$= \left[0+0-0-8\sin^{-1}(-1) - \frac{4}{3}(4)^{3/2}+0\right]$

$= 4\pi - \frac{4}{3}\times 8 = 4\pi - \frac{32}{3}$.

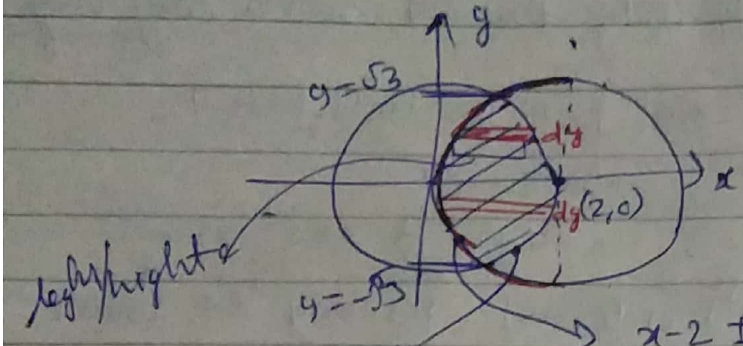
Required Area $= 8\pi - \frac{64}{3}$

[34:06] Example

Area bounded between circle

$x^2+y^2=4$ and $(x-2)^2+y^2=4$.

⇒ draw the graph to look problem easy way.



divide the area into horizontal strip.

Elementary Area = dA = dy (length)

$x-2 = \pm\sqrt{4-y^2}$

$x = \pm\sqrt{4-y^2}$ $\qquad$ $x = 2\pm\sqrt{4-y^2}$

equation of red portion is

$x = 2-\sqrt{4-y^2}$

So length = $[\sqrt{4-y^2} - (2-\sqrt{4-y^2})]$

$\Rightarrow dA = (\sqrt{4-y^2} - (2-\sqrt{4-y^2}))\,dy$

$\Rightarrow A = \int dA = \int_a^b (2\sqrt{4-y^2} - 2)\,dy$

let find value of a and b. which one

So point of intersection.

$(x-2)^2 + 4 - x^2 = 4$

$x^2 - 4x + 4 - x^2 = 0$

$\Rightarrow x = 1, \quad y = \pm\sqrt{3}$

$\Rightarrow \int_{-\sqrt{3}}^{\sqrt{3}} (2\sqrt{4-y^2} - 2)\,dy = 2\int_0^{\sqrt{3}} (2\sqrt{4-y^2} - 2)\,dy$

$\Rightarrow = 4\int_0^{\sqrt{3}} (\sqrt{4-y^2} - 1)\,dy$

$\rightarrow 4\left[\frac{1}{2} y\sqrt{4-y^2} + \frac{1}{2}4\sin^{-1}\left(\frac{y}{2}\right) - y\right]_0^{\sqrt{3}}$

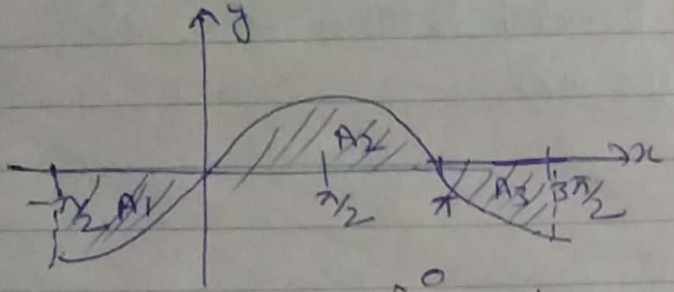
$\Rightarrow 4\left[\frac{1}{2}\sqrt{3} + 2\sin^{-1}\frac{\sqrt{3}}{2} - \sqrt{3}\right] \Rightarrow 4\left(\frac{2\pi}{3} - \frac{1}{2}\sqrt{3}\right)$

$= \boxed{\frac{8}{3}\pi - \frac{2}{3}\sqrt{3}}$

43:03 Another simple example which lie below and above x-axis.

Example :- Area bounded between x-axis, $\sin x$, and $x = -\pi/2$ to $x = 3\pi/2$.

→ lets draw the curve



required area is $A = |A_1| + |A_2| + |A_3|$

$= A = |A_1| + A_2 + |A_3|$

So $A_1 = \int_{-\pi/2}^{0} \sin x\,dx = -\cos x \Big|_{-\pi/2}^{0} = -1$

$A_2 = \int_{0}^{\pi} \sin x\,dx = -\cos x \Big|_{0}^{\pi} = 1 - (-1) = 2.$

$A_3 = \int_{\pi}^{3\pi/2} \sin x\,dx = -\cos x \Big|_{\pi}^{3\pi/2} \Rightarrow -1$

So required area will be $= |-1| + |2| + |-1|$

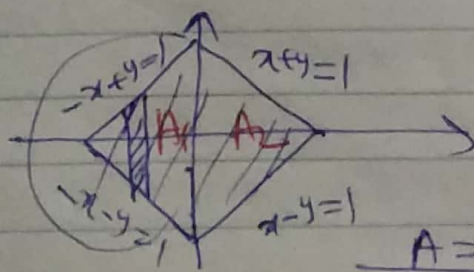
$= \underline{\underline{4}}.$

46:57 Another example.

Find out area bounded by $|x| + |y| = 1$.

this equation represent 4 curves.

$x+y=1, \quad -x+y=1, \quad x-y=1, \quad -x-y=1.$



required area

divide area into the strips

$A = A_1 + A_2$

So for $A_2 \Rightarrow dA = (1 - x - (x-1))\,dx$

$A_2 \Rightarrow \int dA = \int_0^1 (1 - x - (x-1))\,dx + \int_{-1}^{0} (1 + x - (-x-1))\,dx$

$= \int_0^1 2(1-x)\,dx + \int_{-1}^0 2(1+x)\,dx \Rightarrow 2\left[\frac{(1-x)^2}{-2}\right]_0^1 + \left[2\frac{(1+x)^2}{2}\right]_{-1}^0$

$\Rightarrow -2\times\left(-\frac{1}{2}\right) + 2\times\frac{1}{2} = 1+1 = \underline{\underline{2}}.$ Ans

$\boxed{57:05}$ Another example:-

Find out area bounded between the curves.

$y^2 = 4ax$, & $y = mx$.

assume $a>0$, $m>0$.