Definite integral

Lecture - 3

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(00:00) example for solving complicated problem in much simpler way

ex1 $I = \int_{-2}^{2} x^3 \sin^4 x \cos^2 x \, dx$

we will use properties to solve it

we know that

$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ for even function

$= 0$ for odd function.

So lets check

$f(-x) = (-x)^3 \sin^4(-x)\cos^2(-x)$

$= -x^3 \sin^4 x \cos^2 x$

$= -f(x)$ so it is odd function.

hence integration value $I = 0$.

example 2:

$I = \int_{0}^{1} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{1-x}}\,dx$ — (1)

one way is to rationalise it by $\sqrt{x}-\sqrt{1-x}$

but we will do it by property.

we know $\boxed{\int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(a-x)\,dx}$

$I = \int_{0}^{1} \frac{\sqrt{(1-x)}}{\sqrt{1-x}+\sqrt{1-(1-x)}}\,dx = \int_{0}^{1} \frac{\sqrt{1-x}}{\sqrt{1-x}+\sqrt{x}}\,dx$ — (2)

(1) + (2) $\Rightarrow 2I = \int_{0}^{1} \frac{\sqrt{x}+\sqrt{1-x}}{\sqrt{1-x}+\sqrt{x}}\,dx = \int_{0}^{1} dx \Rightarrow 1$

$\Rightarrow I = 1/2$.

Example $I = \int_{-\pi/2}^{\pi/2} \sin^2 x \, dx$ (6:10)

so one way to solve is $\sin^2 x = \frac{1-\cos 2x}{2}$

but we follow diff. method.

$\sin^2(-x) = \sin^2 x \Rightarrow$ it is even.

so

$$I = 2\int_0^{\pi/2} \sin^2 x \, dx \quad -(1)$$

now another property is $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$= 2\int_0^{\pi/2} \sin^2\left(\tfrac{\pi}{2} - x\right) dx$$

$$= 2\int_0^{\pi/2} \cos^2 x \, dx \quad -(2)$$

by (1) + (2) $2I = 2\int_0^{\pi/2} (\sin^2 x + \cos^2 x)\, dx$

$$= 2 \cdot \pi/2$$

$$\Rightarrow \boxed{I = \frac{\pi}{2}}$$ Ans

(8:40) example: $I = \int_0^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x}\right) dx \quad -(1)$

seems very complicated.

but using properties $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$I = \int_0^{\pi/2} \log\left(\frac{4+3\sin(\pi/2 - x)}{4+3\cos(\pi/2 - x)}\right) dx$$

$$I = \int_0^{\pi/2} \log\left(\frac{4+3\cos x}{4+3\sin x}\right) dx \quad -(2)$$

$$2I = \int_0^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x}\right) + \log\left(\frac{4+3\cos x}{4+3\sin x}\right) dx$$

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we know $\log m + \log n = \log mn$

$$= \int_0^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x} \times \frac{4+3\cos x}{4+3\sin x}\right) dx$$

$$= \int_0^{\pi/2} \log(1)\, dx \Rightarrow 0$$

$$\Rightarrow I = 0.$$

Example:

$$I = \int_0^1 x\, e^x\, dx$$

we can use integration by parts.

$$= \boxed{\int uv = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx}$$

$$\Rightarrow x.e^x\Big|_{x=0}^{x=1} - \int_0^1 1.e^x\,dx$$

$$= 1.e^1 - 0.e^0 - e^x\Big|_0^1$$

$$= e - 0 - (e - e^0)$$

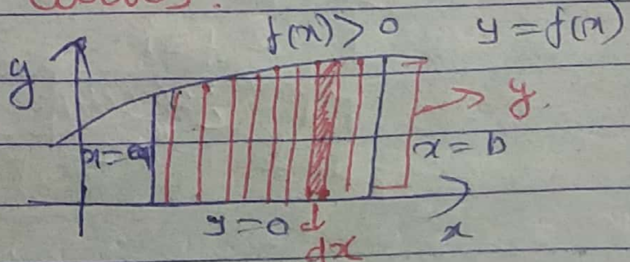
$$= 1$$

$\boxed{13:25}$ Application of definite integration

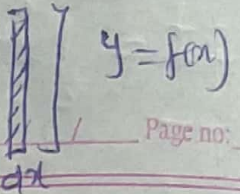
In 1^{st} lecture we have discussed many application.

Area under simple curves.

Case 1:



Area is elementry strip



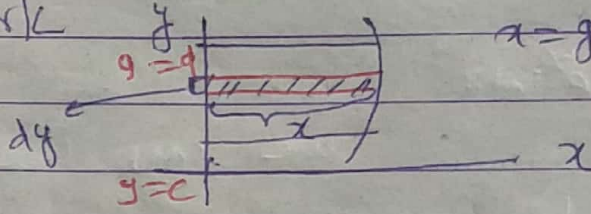
$= y\,dx = dA$

$A = \overset{so}{=} \int_{x=a}^{x=b} dA$ will give us the required Area.

$\Rightarrow$ so $\boxed{A = \int_{x=a}^{b} dA = \int_{a}^{b} y\,dx.}$

Case 2

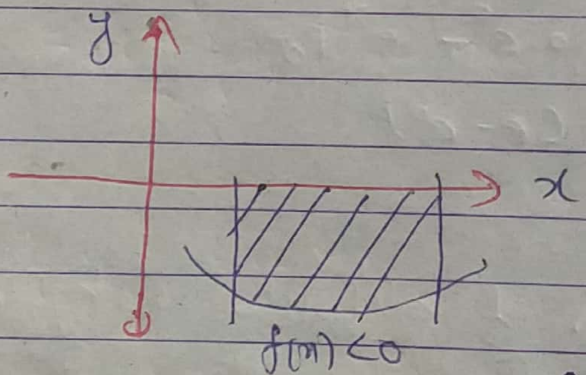
- but there are cases where this method not work



so elementry area is $dA = x\,dy$

$A = \int_{y=c}^{y=d} dA = \int_{y=c}^{y=d} x\,dy.$

Case 3

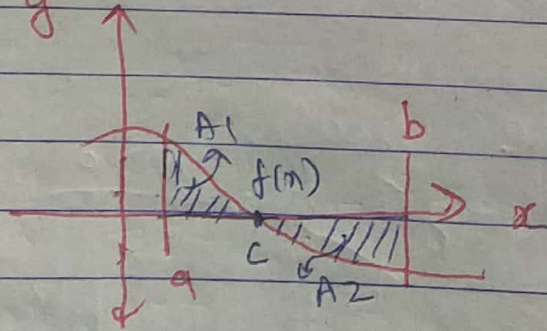


so $A = \int_{a}^{b} f(x)\,dx$

since $A < 0$ as $f(x) < 0.$

so Required Area = $|A|$.

Case 4:



so total Area $A = A_1 + A_2$

$A_1 = \int_{a}^{c} f(x)\,dx$

$A_2 = \int_{c}^{b} f(x)\,dx$

so, $A = A_1 + |A_2|$

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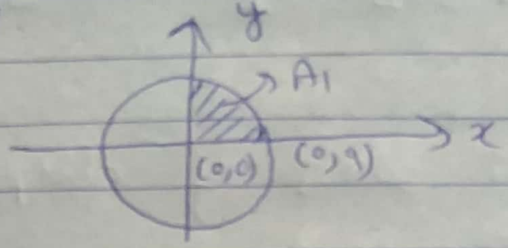
(21:16) Area of simple curve, example.

Ex 1. Area of circle
$x^2+y^2=a^2$



so $\boxed{A = 4 \cdot A_1}$

total Area $A = 4 \times$ (Area of A_1).

and for A_1 we have

$$= 4\int_{x=0}^{x=a} y\,dx$$ and y we have $y = \pm\sqrt{a^2-x^2}$

so upper branch of circle is $y = \sqrt{a^2-x^2}$

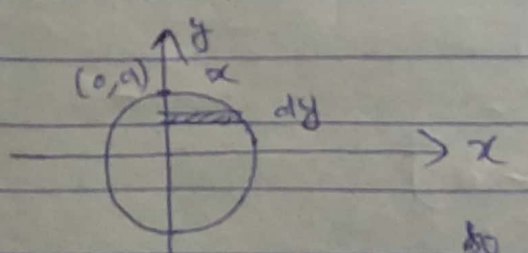
so $\cancel{A_1} = 4\int_0^a \left(\sqrt{a^2-x^2}\right)dx$

$$= 4\left[\frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\sin^{-1}\frac{x}{a}\right]_{x=0}^{x=a}$$

$$= 4\left[\frac{1}{2}a^2\frac{\pi}{2} - 0 - 0\right]$$

$$= \pi a^2$$

Same can be done with horizontal strip



$x^2+y^2=a^2$

$x = \pm\sqrt{a^2-y^2}$

so positive branch is

$x = +\sqrt{a^2-y^2}$

Area of circle $= 4\int_{y=0}^{a} x\,dy$

$$= \int_0^a (\sqrt{a^2-y^2})\, dy$$

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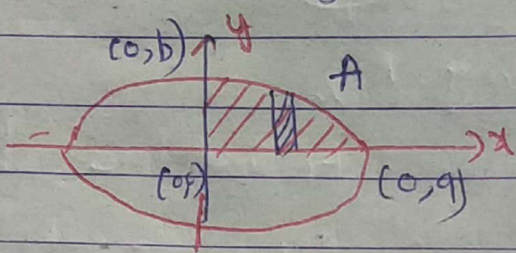
$$= 4\left[\tfrac{1}{2} y\sqrt{a^2-y^2} + \tfrac{1}{2} a^2 \sin^{-1} y/a\right]_{y=0}^{a}$$

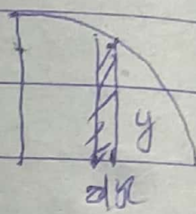
$$= 4\left[\tfrac{1}{2} a^2 \pi/2 - 0 - 0\right] = \pi a^2$$

Example 2: Area of ellips. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $a > b$.



So total Area = 4.(Area of A)

If we use vertical strip method



$$= 4\int_{x=0}^{x=a} y\, dx$$

$$= 4\int_0^a \frac{b}{a}(\sqrt{a^2-x^2})\, dx$$

$y = \pm b\sqrt{1-x^2/a^2}$

$y = \pm \frac{b}{a}\sqrt{a^2-x^2}$

take +ve strip we here

$y = \frac{b}{a}\sqrt{a^2-x^2}$

$$= \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\, dx$$

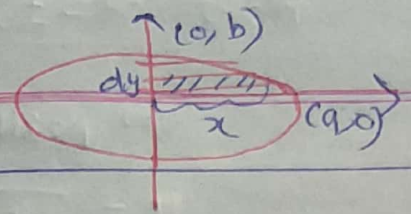
$$= \frac{4b}{a}\left[\tfrac{1}{2} x\sqrt{a^2-x^2} + \tfrac{1}{2} a^2 \sin^{-1}(x/a)\right]_{x=0}^{x=a}$$

$$= \cancel{\frac{4b}{a}\left\{\tfrac{1}{2} x\sqrt{a^2-x^2}\right.}$$

$$= \frac{4b}{a} \times \left[\tfrac{1}{2} a^2 \frac{\pi}{2} - 0 - 0\right]$$

$$= \frac{4b}{a} \times \frac{a^2\pi}{4} \Rightarrow \pi ab.$$

By using horizontal strip



So $\frac{x^2}{a^2} = 1 - \frac{y^2}{b^2}$

$\frac{x}{a} = \pm\sqrt{1-\frac{y^2}{b^2}}$

So $x = \pm a\sqrt{1-\frac{y^2}{b^2}}$

Area of ellips $\Rightarrow 4\int_{y=0}^{b} x\,dy$

$$= 4\int_0^b \frac{a}{b}\left(\sqrt{b^2-y^2}\right)dy$$

$$= 4\frac{a}{b}\left[\frac{1}{2}y\sqrt{b^2-y^2} + \frac{1}{2}b^2\sin^{-1}\left(\frac{y}{b}\right)\right]_0^b$$

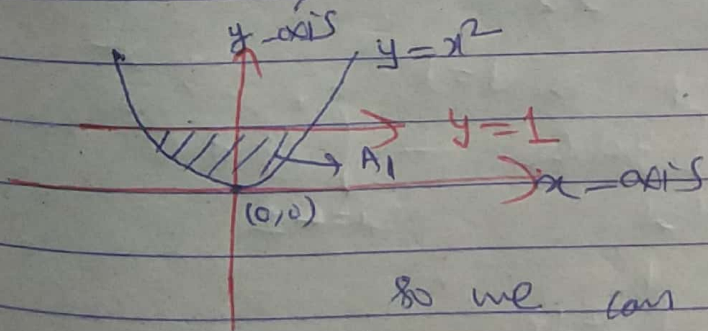
$$= \frac{4a}{b}\left[0+\frac{1}{2}b^2\frac{\pi}{2} - 0 - 0\right]$$

$$= \frac{4\cdot a\cdot \pi b^2}{b\cdot 4} \Rightarrow \pi ab$$

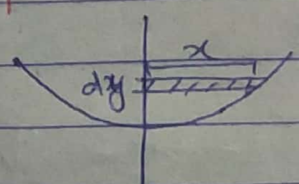
(34:32) Area bounded between a line and a curve.

Example 1

Area bonded betwen $y=1$ and $y=x^2$



So we can use horizontal strip.



So totall area is

$= 2\times(\text{Area } A_1)$.

$$\underline{\underline{So}} = 2\int_{y=0}^{y=1} x\,dy \Rightarrow$$

now $x = \pm\sqrt{y}$

So positive strip is

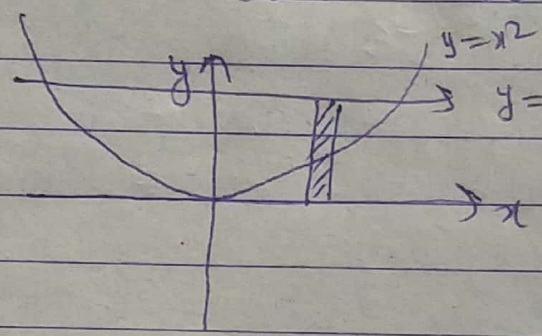
$x = \sqrt{y}$

$$\Rightarrow 2\int_0^1 \sqrt{y}\,dy$$

$$\Rightarrow 2\left[\frac{y^{3/2}}{3/2}\right]_0^1 \Rightarrow 4/3.$$

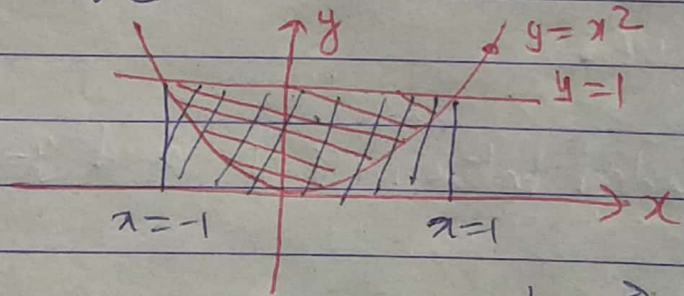
$\Rightarrow$

now let us do it by vertical strips.
(vertical elementary rectangles)



if we take vertical strip there is more not required region.

$\underline{\underline{So}}$ for using vertical strip to find area we use a method. let see.



$$\text{required Area} = \int_{x=-1}^{1} \overset{y=1}{(y)}\,dx - \int_{x=-1}^{1} \overset{x^2}{(y)}\,dx$$

$$= \int_{-1}^{1} 1\,dx - \int_{-1}^{1} x^2\,dx$$

$$= 2 - \left[\frac{x^3}{3}\right]_{-1}^{1} \Rightarrow 2 - 2/3 = 4/3.$$

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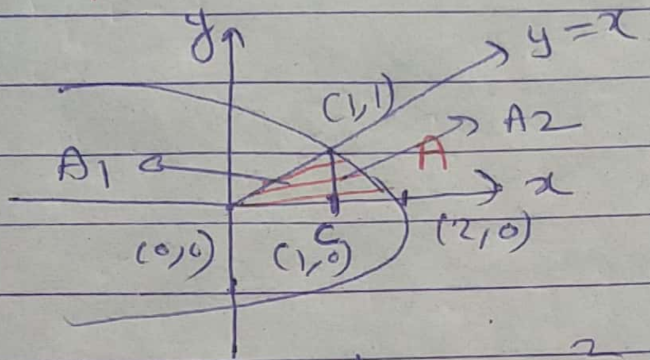
Example Area bounded between $y = x$, $y^2 = 2-x$, and $y = 0$ which is in frist quadrant.



we have find c why $\because x^2 = 2-x$

$$\rightarrow x^2 + x - 2 = 0$$

$$x = -2 \pm 1,$$

So $A = A_1 + A_2$

So $= A_1 = \int_{x=0}^{1} y\,dx + A_2 = \int_{x=1}^{x=2} y\,dx$

$y \rightarrow x$, $y \rightarrow \sqrt{2-x}$

$$A = \int_0^1 x\,dx + \int_1^2 \sqrt{2-x}\,dx$$

$$= \frac{1}{2}x^2\Big|_0^1 + \frac{(2-x)^{3/2}}{-3/2}\Big|_1^2$$

$$= \frac{1}{2} - 0 + 0 - \left(-\frac{2}{3}\right)$$

$$= \frac{1}{2} + \frac{2}{3} \Rightarrow \frac{7}{6} \text{ Ans}$$

(47:30) end