11076CH06

Chapter 6

LINEAR INEQUALITIES

❖*Mathematics is the art of saying many things in many different ways. – MAXWELL*❖

6.1 Introduction

In earlier classes, we have studied equations in one variable and two variables and also solved some statement problems by translating them in the form of equations. Now a natural question arises: 'Is it always possible to translate a statement problem in the form of an equation? For example, the height of all the students in your class is less than 160 cm. Your classroom can occupy atmost 60 tables or chairs or both. Here we get certain statements involving a sign '<' (less than), '>' (greater than), '≤' (less than or equal) and ≥ (greater than or equal) which are known as *inequalities*.

In this Chapter, we will study linear inequalities in one and two variables. The study of inequalities is very useful in solving problems in the field of science, mathematics, statistics, economics, psychology, etc.

6.2 Inequalities

Let us consider the following situations:

(i) Ravi goes to market with ₹ 200 to buy rice, which is available in packets of 1kg. The price of one packet of rice is ₹ 30. If x denotes the number of packets of rice, which he buys, then the total amount spent by him is ₹ $30x$. Since, he has to buy rice in packets only, he may not be able to spend the entire amount of ₹ 200. (Why?) Hence

$$30x < 200 \quad \text{... (1)}$$

Clearly the statement (i) is not an equation as it does not involve the sign of equality.

(ii) Reshma has ₹ 120 and wants to buy some registers and pens. The cost of one register is ₹ 40 and that of a pen is ₹ 20. In this case, if x denotes the number of registers and y, the number of pens which Reshma buys, then the total amount spent by her is ₹ $(40x + 20y)$ and we have

$$40x + 20y \le 120 \quad \text{... (2)}$$

Since in this case the total amount spent may be upto ₹ 120. Note that the statement (2) consists of two statements

$$40x + 20y < 120 \quad \text{... (3)}$$

and

$$40x + 20y = 120 \quad \text{... (4)}$$

Statement (3) is not an equation, i.e., it is an inequality while statement (4) is an equation.

Definition 1 Two real numbers or two algebraic expressions related by the symbol '<', '>', '≤' or '≥' form an *inequality*.

Statements such as (1), (2) and (3) above are inequalities.

$3 < 5$; $7 > 5$ are the examples of *numerical inequalities* while

$x < 5$; $y > 2$; $x \geq 3$, $y \leq 4$ are some examples of *literal inequalities*.

$3 < 5 < 7$ (read as 5 is greater than 3 and less than 7), $3 \leq x < 5$ (read as x is greater than or equal to 3 and less than 5) and $2 < y \leq 4$ are the examples of *double inequalities*.

Some more examples of inequalities are:

$ax + b < 0$ ... (5)

$ax + b > 0$ ... (6)

$ax + b \leq 0$ ... (7)

$ax + b \geq 0$ ... (8)

$ax + by < c$ ... (9)

$ax + by > c$ ... (10)

$ax + by \leq c$ ... (11)

$ax + by \geq c$ ... (12)

$ax^2 + bx + c \leq 0$ ... (13)

$ax^2 + bx + c > 0$ ... (14)

Inequalities (5), (6), (9), (10) and (14) are *strict inequalities* while inequalities (7), (8), (11), (12), and (13) are *slack inequalities*. Inequalities from (5) to (8) are *linear inequalities* in one variable x when $a \neq 0$, while inequalities from (9) to (12) *are linear inequalities in two variables x and y* when $a \neq 0$, $b \neq 0$.

Inequalities (13) and (14) are not linear *(in fact, these are quadratic inequalities in one variable x when* $a \neq 0$).

In this Chapter, we shall confine ourselves to the study of linear inequalities in one and two variables only.

6.3 Algebraic Solutions of Linear Inequalities in One Variable and their Graphical Representation

Let us consider the inequality (1) of Section 6.2, viz, $30x < 200$
Note that here x denotes the number of packets of rice.

Obviously, x cannot be a negative integer or a fraction. Left hand side (L.H.S.) of this inequality is $30x$ and right hand side (RHS) is 200. Therefore, we have

For $x = 0$, L.H.S. = 30 (0) = 0 < 200 (R.H.S.), which is true.

For $x = 1$, L.H.S. = 30 (1) = 30 < 200 (R.H.S.), which is true.

For $x = 2$, L.H.S. = 30 (2) = 60 < 200, which is true.

For $x = 3$, L.H.S. = 30 (3) = 90 < 200, which is true.

For $x = 4$, L.H.S. = 30 (4) = 120 < 200, which is true.

For $x = 5$, L.H.S. = 30 (5) = 150 < 200, which is true.

For $x = 6$, L.H.S. = 30 (6) = 180 < 200, which is true.

For $x = 7$, L.H.S. = 30 (7) = 210 < 200, which is false.

In the above situation, we find that the values of x, which makes the above inequality a true statement, are 0,1,2,3,4,5,6. These values of x, which make above inequality a true statement, are called *solutions* of inequality and the set {0,1,2,3,4,5,6} is called its *solution set*.

Thus, any solution of an inequality in one variable is a value of the variable which makes it a true statement.

We have found the solutions of the above inequality by *trial and error* method which is not very efficient. Obviously, this method is time consuming and sometimes not feasible. We must have some better or systematic techniques for solving inequalities. Before that we should go through some more properties of numerical inequalities and follow them as rules while solving the inequalities.

You will recall that while solving linear equations, we followed the following rules:

Rule 1 Equal numbers may be added to (or subtracted from) both sides of an equation.

Rule 2 Both sides of an equation may be multiplied (or divided) by the same non-zero number.

In the case of solving inequalities, we again follow the same rules except with a difference that in Rule 2, the sign of inequality is reversed (i.e., ‘<‘ becomes ‘>’, ≤’ becomes ‘≥’ and so on) whenever we multiply (or divide) both sides of an inequality by a negative number. It is evident from the facts that

$3 > 2$ while $-3 < -2$,
$-8 < -7$ while $(-8)(-2) > (-7)(-2)$, i.e., $16 > 14$.

Thus, we state the following rules for solving an inequality:

Rule 1 Equal numbers may be added to (or subtracted from) both sides of an inequality without affecting the sign of inequality.

Rule 2 Both sides of an inequality can be multiplied (or divided) by the same positive number. But when both sides are multiplied or divided by a negative number, then the sign of inequality is *reversed.*

Now, let us consider some examples.

Example 1 Solve $30\,x < 200$ when

(i) x is a natural number, (ii) x is an integer.

Solution We are given $30\,x < 200$

or $\frac{30x}{30} < \frac{200}{30}$ (Rule 2), i.e., $x < 20/3$.

(i) When x is a natural number, in this case the following values of x make the statement true.

1, 2, 3, 4, 5, 6.

The solution set of the inequality is {1,2,3,4,5,6}.

(ii) When x is an integer, the solutions of the given inequality are

..., – 3, –2, –1, 0, 1, 2, 3, 4, 5, 6

The solution set of the inequality is {...,–3, –2,–1, 0, 1, 2, 3, 4, 5, 6}

Example 2 Solve $5x - 3 < 3x + 1$ when

(i) x is an integer, (ii) x is a real number.

Solution We have, $5x - 3 < 3x + 1$

or $5x - 3 + 3 < 3x + 1 + 3$ (Rule 1)

or $5x < 3x + 4$

or $5x - 3x < 3x + 4 - 3x$ (Rule 1)

or $2x < 4$

or $x < 2$ (Rule 2)

(i) When x is an integer, the solutions of the given inequality are

..., – 4, – 3, – 2, – 1, 0, 1

(ii) When x is a real number, the solutions of the inequality are given by $x < 2$, i.e., all real numbers x which are less than 2. Therefore, the solution set of the inequality is $x \in (-\infty, 2)$.

We have considered solutions of inequalities in the set of natural numbers, set of integers and in the set of real numbers. Henceforth, unless stated otherwise, we shall solve the inequalities in this Chapter in the set of real numbers.

Example 3 Solve $4x + 3 < 6x + 7$.

Solution We have, $4x + 3 < 6x + 7$

or $4x - 6x < 6x + 4 - 6x$

or $-2x < 4$ or $x > -2$

i.e., all the real numbers which are greater than –2, are the solutions of the given inequality. Hence, the solution set is $(-2, \infty)$.

Example 4 Solve $\frac{5-2x}{3} \le \frac{x}{6} - 5$.

Solution We have

$$\frac{5-2x}{3} \le \frac{x}{6} - 5$$

or $2(5 - 2x) \le x - 30.$

or $10 - 4x \le x - 30$

or $-5x \le -40$, i.e., $x \ge 8$

Thus, all real numbers x which are greater than or equal to 8 are the solutions of the given inequality, i.e., $x \in [8, \infty)$.

Example 5 Solve $7x + 3 < 5x + 9$. Show the graph of the solutions on number line.

Solution We have $7x + 3 < 5x + 9$ or

$2x < 6$ or $x < 3$

The graphical representation of the solutions are given in Fig 6.1.

Fig 6.1

Example 6 Solve $\frac{3x-4}{2} \ge \frac{x+1}{4} - 1$. Show the graph of the solutions on number line.

Solution We have

$$\frac{3x-4}{2} \ge \frac{x+1}{4} - 1$$

or $\frac{3x-4}{2} \ge \frac{x-3}{4}$

or $2(3x - 4) \ge (x - 3)$

or $6x - 8 \geq x - 3$

or $5x \geq 5$ or $x \geq 1$

The graphical representation of solutions is given in Fig 6.2.

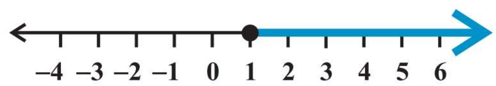

Fig 6.2

Example 7 The marks obtained by a student of Class XI in first and second terminal examination are 62 and 48, respectively. Find the minimum marks he should get in the annual examination to have an average of at least 60 marks.

Solution Let x be the marks obtained by student in the annual examination. Then

$$\frac{62+48+x}{3} \geq 60$$

or $110 + x \geq 180$

or $x \geq 70$

Thus, the student must obtain a minimum of 70 marks to get an average of at least 60 marks.

Example 8 Find all pairs of consecutive odd natural numbers, both of which are larger than 10, such that their sum is less than 40.

Solution Let x be the smaller of the two consecutive odd natural number, so that the other one is $x + 2$. Then, we should have

$x > 10$ ... (1)

and $x + (x + 2) < 40$ ... (2)

Solving (2), we get

$2x + 2 < 40$

i.e., $x < 19$ ... (3)

From (1) and (3), we get

$10 < x < 19$

Since x is an odd number, x can take the values 11, 13, 15, and 17. So, the required possible pairs will be

(11, 13), (13, 15), (15, 17), (17, 19)

EXERCISE 6.1

1. Solve $24x < 100$, when
(i) x is a natural number. (ii) x is an integer.

2. Solve $-12x > 30$, when
(i) x is a natural number. (ii) x is an integer.

3. Solve $5x - 3 < 7$, when
(i) x is an integer. (ii) x is a real number.

4. Solve $3x + 8 > 2$, when
(i) x is an integer. (ii) x is a real number.

Solve the inequalities in Exercises 5 to 16 for real x.

5. $4x + 3 < 5x + 7$

6. $3x - 7 > 5x - 1$

7. $3(x - 1) \leq 2(x - 3)$

8. $3(2 - x) \geq 2(1 - x)$

9. $x + \frac{x}{2} + \frac{x}{3} < 11$

10. $\frac{x}{3} > \frac{x}{2} + 1$

11. $\frac{3(x-2)}{5} \leq \frac{5(2-x)}{3}$

12. $\frac{1}{2}\left(\frac{3x}{5} + 4\right) \geq \frac{1}{3}(x-6)$

13. $2(2x + 3) - 10 < 6(x - 2)$

14. $37 - (3x + 5) \geq 9x - 8(x - 3)$

15. $\frac{x}{4} < \frac{(5x-2)}{3} - \frac{(7x-3)}{5}$

16. $\frac{(2x-1)}{3} \geq \frac{(3x-2)}{4} - \frac{(2-x)}{5}$

Solve the inequalities in Exercises 17 to 20 and show the graph of the solution in each case on number line

17. $3x - 2 < 2x + 1$

18. $5x - 3 \geq 3x - 5$

19. $3(1 - x) < 2(x + 4)$

20. $\frac{x}{2} \geq \frac{(5x-2)}{3} - \frac{(7x-3)}{5}$

21. Ravi obtained 70 and 75 marks in first two unit test. Find the minimum marks he should get in the third test to have an average of at least 60 marks.

22. To receive Grade 'A' in a course, one must obtain an average of 90 marks or more in five examinations (each of 100 marks). If Sunita's marks in first four examinations are 87, 92, 94 and 95, find minimum marks that Sunita must obtain in fifth examination to get grade 'A' in the course.

23. Find all pairs of consecutive odd positive integers both of which are smaller than 10 such that their sum is more than 11.

24. Find all pairs of consecutive even positive integers, both of which are larger than 5 such that their sum is less than 23.