Q ⇒ Sol^n of a diff equation.

⇒ General sol^n (complete primitive)

A relation in X and Y satisfying a given

X ↓ independent variable

diff eqn and involving exactly the same no of arbitrary constant as the order of the diff equation.

→ Particular Solution.
A sol^n obtained by assigning values to the arbitrary constant found in general sol^n.

→ Singular sol^n.
A sol^n which cannot be obtained from the general sol^n but still is a sol^n of ~~the~~ given diff eqn. ~~geom~~ geometrically ~~it~~ singular sol^n is as an envelope to the general sol^n.
for ex:

$$\left(\frac{dy}{dx}\right)^2 - 4y = 0$$

$y = (x+c)^2$ ← general sol^n

but $y = 0$ also satisfies
↳ singular sol^n of this diff equation.

⇒ variable seperiable Method.

$$\int f(x)\,dx = \int g(y)\,dy.$$

① Solve

① $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0.$

$$\int \frac{\sec^2 x}{\tan x} dx = \int -\frac{\sec^2 y}{\tan y} dy$$

$\ln \tan x + \ln c = -\ln \tan y$

$k \tan x = \tan y$

$\boxed{y = \tan^{-1}(k \tan x)} \to$ general solⁿ

$|\tan x \tan y| = k$

Q Solve: $\frac{dy}{dx} = 1 + x + y + xy$

find particular solⁿ which passes thro (0,2)

$\frac{dy}{dx} = (1+y)(1+x)$ $\boxed{y = 3e^{x(1+\frac{x}{2})} - 1}$

$\ln(1+y) = (x + x^2 + c)$

$1 + y = k e^{x+x^2}$

$\boxed{y = k e^{x + \frac{x^2}{2}} - 1}$ $k = 3$

Solve $\sqrt{1+x^2+y^2+x^2y^2} + xy\frac{dy}{dx} = 0$

$(1+x^2)^{1/2}(1+y^2)^{1/2} = -xy\frac{dy}{dx}$

$\int \frac{\sqrt{1+x^2}}{x} dx = \int -\frac{y}{\sqrt{1+y^2}} dy$

$\frac{\sec^2 x \, dx}{\tan x}$

$\frac{1}{\sin x \cos^2 x}$

$\frac{1}{2}\int \frac{\sqrt{1+t}}{t}\,dt.\quad t=x^2$ $\qquad \frac{1+t}{t\sqrt{1+t}}$ $\qquad \frac{1}{\sqrt{1+t}} + \frac{1}{\sqrt{1+t}}$

$$\int \frac{1+x^2}{x\sqrt{1+x^2}}\,dx = -\frac{1}{2}\int \frac{2y}{\sqrt{1+y^2}}\,dy$$

$$= -\sqrt{1+y^2}$$

$$-\left[\sqrt{1+x^2} + \frac{1}{2}\ln\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right|\right] = \sqrt{1+y^2} + c$$

Q. If $e^{\frac{dy}{dx}} = x+1$ and $y(0)=3$ then find $y(e-1)$.

$$\frac{dy}{dx} = \ln(x+1)$$

$$\int dy = \int \ln(x+1)\,dx$$

$$y = (x+1)(\ln(x+1)-1) + c$$

$$3 = -1 + c$$

$$c = 4$$

$$y = (x+1)(\ln(x+1)-1) + 4$$

$$\boxed{y(e-1) = 4}$$

$\Rightarrow$ diff eqn reducable to variable seperable.

$$\frac{dy}{dx} = f(ax+by+c)$$

$$ax+by+c = v$$

$$a + b\frac{dy}{dx} = \frac{dv}{dx}.$$

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solve

$(4x+y+1) = v$

$4 + \frac{dy}{dx} = \frac{dv}{dx}$

(1) $\frac{dy}{dx} = (4x+y+1)^2$

$\frac{dv}{dx} = v^2 + 4$

$\frac{1}{2}\tan^{-1}\left(\frac{v}{2}\right) = x + c.$

$\tan^{-1}\left(\frac{4x+y+1}{2}\right) = 2x + c$

$\boxed{4x+y+1 = 2\tan(2x+c)}$

(2) $\frac{dy}{dx} = \cos(x+y) + \sin(x+y).$

$x+y = v$

$1 + \frac{dy}{dx} = \frac{dv}{dx}$

$\frac{dv}{dx} = \cos v + \sin v + 1$

$\int \frac{dv}{\cos v + \sin v + 1} = \int dx$

$\frac{2 \times \frac{1}{2}\sec^2 v/2}{1 - \tan^2\frac{v}{2} + 2\tan\frac{v}{2} + 1 + \tan^2\frac{v}{2}}$

$\frac{1}{1 - \cancel{t^2} + 2t + 1 + \cancel{t^2}}$

$\frac{\cancel{2}}{\cancel{2}} \ln\left(1 + \tan\frac{v}{2}\right) = x + c$

$1 + \tan\left(\frac{x+y}{2}\right) = ke^x$

$\boxed{\tan\left(\frac{x+y}{2}\right) = ke^x - 1}$

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Solve

③ $\frac{dy}{dx} = \frac{2x-y+2}{2y-4x+1}$

$2x-y = v$

$2 - \frac{dy}{dx} = \frac{dv}{dx}$

$\frac{dv}{dx} = -\frac{(v+2)}{1-2v} + 2$

$\frac{dv}{dx} = \frac{v+\cancel{2}+4v-\cancel{2}}{2v-1}$

$\frac{dv\,(2v-1)}{5v} = dx$

$dv\left(\frac{2}{5} - \frac{1}{5v}\right) = dx.$

$\frac{2v}{5} - \frac{1}{5}\ln v = x + c \Rightarrow \frac{2}{5}(2x-y) - \frac{1}{5}\ln(2x-y)$

$4x - 2y - \ln(2x-y) = 5x + c.$

$$\boxed{-\ln(2x-y) = x + 2y + c}$$

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$\Rightarrow$ homogeneous diff eqn

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \quad \text{or} \quad \frac{dy}{dx} = F\left(\frac{x}{y}\right)$$

If $f(x, y)$. $\quad x \to \lambda x$

$y \to \lambda y$

$f(\lambda x, \lambda y) = \lambda^{\circ} f(x, y)$

then it is homogeneous eqn

put $y = vx$ or $x = vy$.

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

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solve

① $$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy} = \frac{1}{2}\frac{y}{x} - \frac{x}{2y}$$

$$\frac{y}{x} = v$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx} = \frac{v}{2} - \frac{1}{2v}$$

$$\frac{v}{2} + \frac{1}{2v} = x\frac{dv}{dx}$$

$$\frac{v^2+1}{2v} = \frac{dv}{dx}\, x$$

$$\int \frac{dx}{x} = \int \frac{dv\; 2v}{(v^2+1)}$$

$\ln xk = \ln(v^2+1)$

$xk = v^2+1$

$xc = \frac{y^2}{x^2} + 1$

$$\boxed{y^2 + x^2 = x^3 c}$$

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(7) Solve

$$x\frac{dy}{dx} = \sqrt{x^2 - y^2} + y$$

$$\frac{dy}{dx} = \sqrt{1-\left(\frac{y}{x}\right)^2} + \frac{y}{x}$$

$$\cancel{v} + \frac{dv}{dx} = \sqrt{1-v^2} + \cancel{v}$$

$$\int \frac{dv}{\sqrt{1-v^2}} = \int dx \qquad \boxed{\cancel{y = x\sin(x+c)}}$$

$$\sin^{-1}(v) = \ln x + c. \qquad \boxed{y = x\sin(\ln xc)}$$

$$\frac{y}{x} = \sin(x+c)$$

$y = x$ also satisfies.

$$\frac{y}{x} = \sin(\ln xc)$$

it got missed by?

here we devided by x and $\sqrt{1-v^2}$ but they can be zero, this can lead to loss of soln.

$$1 - v^2 = 0 \qquad y = \pm x.$$

looking at eqn $y = x$, $y = -x$ is also soln.

but $x = 0$ is not satisfying

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Q $x^3 \frac{dy}{dx} = y^3 + y^2\sqrt{y^2 - x^2}$

$y = 0$ satisfies.

$$\frac{dy}{dx} = \left(\frac{y}{x}\right)^3 + \left(\frac{y}{x}\right)^2\sqrt{\left(\frac{y}{x}\right)^2 - 1}$$

$$v + x\frac{dv}{dx} = v^3 + v^2\sqrt{v^2-1}$$

stuck on Integration

$$\int \frac{dv}{v^3 + v^2\sqrt{v^2-1} - v} = \int \frac{dx}{x}$$

$\boxed{\text{Redo this}}$

$\boxed{y^2 cx(cx-2) + x^2 = 0}$

$$\frac{x}{y} = v$$

$$\frac{dx}{dy} = v + y\frac{dv}{dy}$$

$$\left(\frac{x}{y}\right)^3\left(\frac{dy}{dx}\right) = 1 + 1\sqrt{1 - \left(\frac{x}{y}\right)^2}$$

$$\frac{v^3}{v + y\frac{dv}{dy}} = 1 + \sqrt{1-v^2}$$

$$\frac{v^3}{1+\sqrt{1-v^2}} = v + y\frac{dv}{dy}$$

$$\frac{v^3 - v + v\sqrt{1-v^2}}{1+\sqrt{1-v^2}} = y\frac{dv}{dy}$$

$$\int \frac{dy}{y} = \int \frac{(1+\sqrt{1-v^2})\,dv}{v^3 - v + v\sqrt{1-v^2}}$$

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Q $(1+2e^{x/y})\,dx + 2e^{x/y}\left(1-\frac{x}{y}\right)dy = 0.$

$$\frac{dy}{dx}\left(2e^{x/y}\left(1-\frac{x}{y}\right)\right) = -\left(1+2e^{x/y}\right)$$

$$v + x\frac{dv}{dx} = \frac{1+2e^{v}}{(v-1)\,2e^{v}}$$

$$x\frac{dv}{dx} = \frac{1+2e^{v} - v(v-1)(2e^{v})}{(v-1)\,2e^{v}}$$

$$\frac{(v-1)\,2e^{v}\;dv}{1+2e^{v} - (v)(v-1)\,2e^{v}} = \frac{dx}{x}$$

$~~2e^{v} - 2v^2 e^{v} + v2e^{v} + 1~~$

$2e^{v} - 4ve^{v} - 2v^3e^{v} +$

$$\boxed{x + 2ye^{x/y} = C}$$

⇒ eqn reducable to homogeneous.

Example: $\frac{dy}{dx} = \frac{x+2y+3}{2x+3y+4}$

$x = X + h \quad , \quad y = Y + k.$

$$\frac{dY}{dX} = \frac{dy}{dx}$$

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$$\frac{dY}{dX} = \frac{(X+h) + 2(Y+k) + 3}{2(X+h) + 3(Y+k) + 4}$$

$h + 2k + 3 = 0$ $\quad h = 1 \quad k = -2$

$2h + 3k + 4 = 0$ $\quad X = x - 1 \quad Y = y + 2$

$$\frac{dY}{dX} = \frac{X + 2Y}{2X + 3Y}$$

here we have just shifted the origin to the point of intersection of lines

$$Y = vX$$

$$\frac{dY}{dX} = v + X\frac{dv}{dX}$$

$$\frac{X\,dv}{dX} = \frac{1 + 2v - 2v - 3v^2}{2 + 3v} \qquad \int \frac{2+3v}{1-3v^2}\,dv = \int \frac{dx}{v}$$

Q $\frac{dy}{dx} = \frac{(x+y)^2}{(x+2)(y-2)}$ $\qquad x = X - 2$

$y = Y + 2.$

$$\frac{dy}{dx} = \frac{(X+Y)^2}{XY} \qquad \frac{1}{2}\int \frac{2v+1-1}{1+2v}\,dv = \int \frac{dx}{dx}$$

$$\frac{\left[1 + \left(\frac{Y}{\cancel{X}}\right)\right]^2}{\frac{\cancel{Y}}{X}}$$

$$v - \frac{\ln(1+2v)}{2} = \ln cx$$

$$v + \frac{X\,dv}{dX} = \frac{(1+v)^2}{v}$$

$$\frac{1+2v}{v} = \frac{X\,dv}{dX}$$

$$\frac{Y}{X} - \frac{\ln}{2}\left(1 + 2\frac{Y}{X}\right) = \ln cX$$

$$\boxed{e^{\frac{y-2}{x+2}} = c(x+1)\left(1 + 2\frac{y-2}{x+1}\right)^{1/2}}$$

⇒ linear diff equation

The linear diff eqn are those eqn in which the dependent variable (x) the independent variable and its first derivative appears only in first degree and are not multiplied together. The coefficient must constant or function of x.

Ex:
$$\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right) + y = 0 \checkmark$$

$$\frac{d^2y}{dx^2} + y\left(\frac{dy}{dx}\right) + x = 0 \times$$

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + y^2 = 0 \times$$

* Linear diff eqn of first order is of the form.

$$\frac{dy}{dx} + Py = Q.$$

where P and Q are function of x or constant.

Now how to solve it
multiply by $e^{\int P\,dx}$

$$e^{\int P dx}\frac{dy}{dx} + e^{\int P dx} P y = e^{\int P dx} Q$$

$$\int \frac{d}{dx}\left(e^{\int P dx} y\right) = \int e^{\int P dx} Q$$

$$y\, e^{\int P dx} = \int \left(e^{\int P dx} Q\right) dx$$

here $e^{\int P dx}$ is called integrating factor
so first identify L eqn then

$$y\,(I.F) = \int (I.F)\, Q\, dx$$

All depends on dependent and independent variable.

Q $$\frac{dx}{dy} + Rx = S$$

$$x\,(I.F) = \int (I.F)\, dy \quad (I.F) = e^{\int R\,dy}$$

$\frac{x-(1+x)}{(1+x)}$ $\left(\frac{-1}{x^2}\right)$

Q Solve

x

$\frac{1}{x}+x$

$$\frac{dy}{dx} + \left(\frac{x}{1+x}\right) y = \frac{1-x}{1+x}$$

$\frac{-1}{x^2}+1$ $\frac{x^2-1}{x}$

$$e^{-\int \frac{x+1}{1+x} dx}$$

$$e^{-x+\ln(1+x)} \Rightarrow y e^{\ln(1+x)-x} = \int e^{\ln(1+x)-x}\left(\frac{1-x}{1+x}\right)$$

$$y(x+1)e^{-x} = \int e^{-x}(1-x)$$

$$y(x+1) = e^{-x}(x)/e^{-x}$$

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Solve : $(1+y^2)\,dx = (\tan^{-1}(y) - x)\,dy$.

$$(1+y^2)\frac{dx}{dy} + x = \tan^{-1}(y).$$

$$\frac{dx}{dy} + \frac{x}{(1+y^2)} = \frac{\tan^{-1}(y)}{(1+y^2)}.$$

$$x e^{\tan^{-1}y} = \int t e^t$$

$$x e^{\tan^{-1}y} = \tan^{-1}(y)\, e^{\tan^{-1}y} - e^{\tan^{-1}(y)} + c$$

$$\boxed{x = (\tan^{-1}(y) - 1) + c e^{-\tan^{-1}(y)}}$$

Q If y_1, y_2 are two solⁿ of diff eqn

$$\frac{dy}{dx} + P(x)\,y = Q(x).$$

then show hat general solⁿ is of he form $(y = \alpha y_1 + \beta y_2)$ also find relation btw α and β.

$$y_1 e^{\int P(x)} = \int e^{\int P(x)} Q(x) + C$$

$$y_2 e^{\int P(x)} = \int e^{\int P(x)} Q(x) + C'$$

$$y = \frac{\int e^{\int P(x)} Q(x)}{e^{\int P(x)}} + C'' e^{-\int P(x)dx}$$

$$y = \frac{y_1 - C e^{-\int P(x)} + y_2 - C' e^{-\int P(x)}}{2} + c$$

$$y_1 - ce^{-\int P(x)} = \frac{\int e^{\int P(x)} Q(x)}{e^{\int P(x)}} = y_2 - c'e^{-\int P(x)}$$

$\frac{dy}{dx} + P(x)y = Q(x)$ —(1)

$\frac{dy_2}{dx} + P(x)y_2 = Q(x)$ —(3)

$\frac{dy_1}{dx} + P(x)y_1 = Q(x)$ —(2)

from (1) and (2)

$\frac{d(y-y_1)}{dx} + P(x)(y-y_1) = 0$ —(4)

from (2) and (3).

$\frac{d(y_1-y_2)}{dx} + P(x)(y_1-y_2) = 0$ —(5)

$\frac{d(y-y_1)}{y-y_1} = -P(x)\,dx$

$\frac{d(y_1-y_2)}{(y_1-y_2)} = -P(x)\,dx$

$\ln(y-y_1) = \ln(y_1-y_2) + \ln c$

$y - y_1 = cy_1 - cy_2$

$\boxed{y = (1+c)y_1 - cy_2}$ hence proved

$y = \alpha y_1 + \beta y_2$

$\alpha = 1+c$

$\beta = -c$

$\boxed{\alpha + \beta = 1}$

⇒ equation reducible to linear differential eqn.

Bernoulli's equation

$f'(y)\frac{dy}{dx} + f(y)P(x) = Q(x)$

$f(y) = t$. $f'(y)\frac{dy}{dx} = \frac{dt}{dx}$.

$\frac{dt}{dx} + P(x)t = Q(x)$.

Q Solu.: $\frac{dy}{dx} - y\tan x = -y^2\sec x$

$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}\tan x = -\sec x$.

$\frac{1}{y} = t$ $\frac{-1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$

$+\frac{dt}{dx} + t\tan x = +\sec x$

$t\sec x = \int \sec^2 x + c$

$\frac{1}{y}\sec x = \tan x + c$

$\boxed{\sec x = y(\tan x + c)}$

Q $\sin y\frac{dy}{dx} = \cos y(1 - x\cos y)$.

$\sin y\frac{dy}{dx} + x\cos^2 y = \cos y$

$\cos y = t$ $-\sin y\frac{dy}{dx} = \frac{dt}{dx}$

$+\frac{dt}{dx} + tx = 1$

$t e^{\frac{x^2}{2}}$