EXERCISE 9.3

In each of the Exercises 1 to 5, form a differential equation representing the given family of curves by eliminating arbitrary constants a and b.

1. $\frac{x}{a}+\frac{y}{b}=1$ **2.** $y^2 = a\,(b^2 - x^2)$ **3.** $y = a\,e^{3x} + b\,e^{-2x}$

4. $y = e^{2x}\,(a + bx)$ **5.** $y = e^{x}\,(a \cos x + b \sin x)$

6. Form the differential equation of the family of circles touching the y-axis at origin.

7. Form the differential equation of the family of parabolas having vertex at origin and axis along positive y-axis.

8. Form the differential equation of the family of ellipses having foci on y-axis and centre at origin.

9. Form the differential equation of the family of hyperbolas having foci on x-axis and centre at origin.

10. Form the differential equation of the family of circles having centre on y-axis and radius 3 units.

11. Which of the following differential equations has $y = c_1\,e^{x} + c_2\,e^{-x}$ as the general solution?

(A) $\frac{d^2y}{dx^2}+y=0$ (B) $\frac{d^2y}{dx^2}-y=0$ (C) $\frac{d^2y}{dx^2}+1=0$ (D) $\frac{d^2y}{dx^2}-1=0$

12. Which of the following differential equations has $y = x$ as one of its particular solution?

(A) $\frac{d^2y}{dx^2}-x^2\frac{dy}{dx}+xy=x$ (B) $\frac{d^2y}{dx^2}+x\frac{dy}{dx}+xy=x$

(C) $\frac{d^2y}{dx^2}\quad x^2\frac{dy}{dx}\quad xy\quad 0$ (D) $\frac{d^2y}{dx^2}+x\frac{dy}{dx}+xy=0$

9.5. Methods of Solving First Order, First Degree Differential Equations

In this section we shall discuss three methods of solving first order first degree differential equations.

9.5.1 *Differential equations with variables separable*

A first order-first degree differential equation is of the form

$$\frac{dy}{dx} = F(x, y) \qquad \dots (1)$$

If F (x, y) can be expressed as a product g (x) $h(y)$, where, $g(x)$ is a function of x and $h(y)$ is a function of y, then the differential equation (1) is said to be of variable separable type. The differential equation (1) then has the form

$$\frac{dy}{dx} = h(y) \cdot g(x) \qquad \text{... (2)}$$

If $h(y) \neq 0$, separating the variables, (2) can be rewritten as

$$\frac{1}{h(y)} \, dy = g(x) \, dx \qquad \text{... (3)}$$

Integrating both sides of (3), we get

$$\int \frac{1}{h(y)} \, dy = \int g(x) \, dx \qquad \text{... (4)}$$

Thus, (4) provides the solutions of given differential equation in the form

$$H(y) = G(x) + C$$

Here, H (y) and G (x) are the anti derivatives of $\frac{1}{h(y)}$ and $g(x)$ respectively and C is the arbitrary constant.

Example 9 Find the general solution of the differential equation $\frac{dy}{dx} = \frac{x+1}{2-y}, (y \neq 2)$

Solution We have

$$\frac{dy}{dx} = \frac{x+1}{2-y} \qquad \text{... (1)}$$

Separating the variables in equation (1), we get

$$(2 - y) \, dy = (x + 1) \, dx \qquad \text{... (2)}$$

Integrating both sides of equation (2), we get

$$\int (2-y) \, dy = \int (x+1) \, dx$$

or $$2y - \frac{y^2}{2} = \frac{x^2}{2} + x + C_1$$

or $$x^2 + y^2 + 2x - 4y + 2 C_1 = 0$$

or $$x^2 + y^2 + 2x - 4y + C = 0, \text{ where } C = 2C_1$$

which is the general solution of equation (1).

Example 10 Find the general solution of the differential equation $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$.

Solution Since $1 + y^2 \neq 0$, therefore separating the variables, the given differential equation can be written as

$$\frac{dy}{1+y^2} = \frac{dx}{1+x^2} \quad \text{... (1)}$$

Integrating both sides of equation (1), we get

$$\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$$

or $$\tan^{-1} y = \tan^{-1} x + C$$

which is the general solution of equation (1).

Example 11 Find the particular solution of the differential equation $\frac{dy}{dx} = -4xy^2$ given that $y = 1$, when $x = 0$.

Solution If $y \neq 0$, the given differential equation can be written as

$$\frac{dy}{y^2} = -4x\, dx \quad \text{... (1)}$$

Integrating both sides of equation (1), we get

$$\int \frac{dy}{y^2} = -4\int x\, dx$$

or $$-\frac{1}{y} = -2x^2 + C$$

or $$y = \frac{1}{2x^2 - C} \quad \text{... (2)}$$

Substituting $y = 1$ and $x = 0$ in equation (2), we get, $C = -1$.

Now substituting the value of C in equation (2), we get the particular solution of the given differential equation as $y = \frac{1}{2x^2+1}$.

Example 12 Find the equation of the curve passing through the point (1, 1) whose differential equation is $x\, dy = (2x^2 + 1)\, dx$ $(x \neq 0)$.

Solution The given differential equation can be expressed as

$$dy^* = \frac{2x^2 \quad 1}{x}\, dx^*$$

or $$dy = \left(2x + \frac{1}{x}\right) dx \qquad \text{... (1)}$$

Integrating both sides of equation (1), we get

$$\int dy = \int \left(2x + \frac{1}{x}\right) dx$$

or $$y = x^2 + \log|x| + C \qquad \text{... (2)}$$

Equation (2) represents the family of solution curves of the given differential equation but we are interested in finding the equation of a particular member of the family which passes through the point (1, 1). Therefore substituting $x = 1$, $y = 1$ in equation (2), we get C = 0.

Now substituting the value of C in equation (2) we get the equation of the required curve as $y = x^2 + \log|x|$.

Example 13 Find the equation of a curve passing through the point (–2, 3), given that the slope of the tangent to the curve at any point (x, y) is $\frac{2x}{y^2}$.

Solution We know that the slope of the tangent to a curve is given by $\frac{dy}{dx}$.

so, $$\frac{dy}{dx} = \frac{2x}{y^2} \qquad \text{... (1)}$$

Separating the variables, equation (1) can be written as

$$y^2\, dy = 2x\, dx \qquad \text{... (2)}$$

Integrating both sides of equation (2), we get

$$\int y^2 dy = \int 2x\, dx$$

or $$\frac{y^3}{3} = x^2 + C \qquad \text{... (3)}$$

* The notation $\frac{dy}{dx}$ due to Leibnitz is extremely flexible and useful in many calculation and formal transformations, where, we can deal with symbols *dy* and *dx* exactly as if they were ordinary numbers. By treating *dx* and *dy* like separate entities, we can give neater expressions to many calculations.

Refer: Introduction to Calculus and Analysis, volume-I page 172, By Richard Courant, Fritz John Spinger – Verlog New York.

Substituting $x = -2$, $y = 3$ in equation (3), we get C = 5.

Substituting the value of C in equation (3), we get the equation of the required curve as

$$\frac{y^3}{3} = x^2 + 5 \quad \text{or} \quad y = (3x^2 + 15)^{\frac{1}{3}}$$

Example 14 In a bank, principal increases continuously at the rate of 5% per year. In how many years Rs 1000 double itself?

Solution Let P be the principal at any time t. According to the given problem,

$$\frac{dp}{dt} = \left(\frac{5}{100}\right) \times \text{P}$$

or
$$\frac{dp}{dt} = \frac{\text{P}}{20} \quad \text{... (1)}$$

separating the variables in equation (1), we get

$$\frac{dp}{\text{P}} = \frac{dt}{20} \quad \text{... (2)}$$

Integrating both sides of equation (2), we get

$$\log \text{P} = \frac{t}{20} + \text{C}_1$$

or
$$\text{P} = e^{\frac{t}{20}} \cdot e^{\text{C}_1}$$

or
$$\text{P} = \text{C}\, e^{\frac{t}{20}} \quad (\text{where } e^{\text{C}_1} = \text{C}) \quad \text{... (3)}$$

Now $\quad$ P = 1000, when $t = 0$

Substituting the values of P and t in (3), we get C = 1000. Therefore, equation (3), gives

$$\text{P} = 1000\, e^{\frac{t}{20}}$$

Let t years be the time required to double the principal. Then

$$2000 = 1000\, e^{\frac{t}{20}} \Rightarrow t = 20 \log_e 2$$

EXERCISE 9.4

For each of the differential equations in Exercises 1 to 10, find the general solution:

1. $\frac{dy}{dx} = \frac{1 - \cos x}{1 + \cos x}$

2. $\frac{dy}{dx} = \sqrt{4 - y^2} \quad (-2 < y < 2)$

3. $\dfrac{dy}{dx} + y = 1 \; (y \neq 1)$

4. $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$

5. $(e^x + e^{-x}) \, dy - (e^x - e^{-x}) \, dx = 0$

6. $\dfrac{dy}{dx} = (1 + x^2)(1 + y^2)$

7. $y \log y \, dx - x \, dy = 0$

8. $x^5 \dfrac{dy}{dx} = -y^5$

9. $\dfrac{dy}{dx} = \sin^{-1} x$

10. $e^x \tan y \, dx + (1 - e^x) \sec^2 y \, dy = 0$

For each of the differential equations in Exercises 11 to 14, find a particular solution satisfying the given condition:

11. $(x^3 + x^2 + x + 1)\dfrac{dy}{dx} = 2x^2 + x$; $y = 1$ when $x = 0$

12. $x(x^2 - 1)\dfrac{dy}{dx} = 1$; $y = 0$ when $x = 2$

13. $\cos\left(\dfrac{dy}{dx}\right) = a \quad (a \in \mathbf{R})$; $y = 2$ when $x = 0$

14. $\dfrac{dy}{dx} = y \tan x$; $y = 1$ when $x = 0$

15. Find the equation of a curve passing through the point (0, 0) and whose differential equation is $y' = e^x \sin x$.

16. For the differential equation $xy\dfrac{dy}{dx} = (x + 2)(y + 2)$, find the solution curve passing through the point (1, –1).

17. Find the equation of a curve passing through the point (0, –2) given that at any point (x, y) on the curve, the product of the slope of its tangent and y coordinate of the point is equal to the x coordinate of the point.

18. At any point (x, y) of a curve, the slope of the tangent is twice the slope of the line segment joining the point of contact to the point (– 4, –3). Find the equation of the curve given that it passes through (–2, 1).

19. The volume of spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of balloon after t seconds.

20. In a bank, principal increases continuously at the rate of r% per year. Find the value of r if Rs 100 double itself in 10 years ($\log_e 2 = 0.6931$).

21. In a bank, principal increases continuously at the rate of 5% per year. An amount of Rs 1000 is deposited with this bank, how much will it worth after 10 years ($e^{0.5} = 1.648$).

22. In a culture, the bacteria count is 1,00,000. The number is increased by 10% in 2 hours. In how many hours will the count reach 2,00,000, if the rate of growth of bacteria is proportional to the number present?

23. The general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$ is

(A) $e^x + e^{-y} = C$ (B) $e^x + e^y = C$

(C) $e^{-x} + e^y = C$ (D) $e^{-x} + e^{-y} = C$

9.5.2 *Homogeneous differential equations*

Consider the following functions in x and y

$F_1(x, y) = y^2 + 2xy$, $F_2(x, y) = 2x - 3y$,

$F_3(x, y) = \cos\left(\frac{y}{x}\right)$, $F_4(x, y) = \sin x + \cos y$

If we replace x and y by λx and λy respectively in the above functions, for any nonzero constant λ, we get

$$F_1(\lambda x, \lambda y) = \lambda^2 (y^2 + 2xy) = \lambda^2 F_1(x, y)$$

$$F_2(\lambda x, \lambda y) = \lambda (2x - 3y) = \lambda F_2(x, y)$$

$$F_3(\lambda x, \lambda y) = \cos\left(\frac{\lambda y}{\lambda x}\right) = \cos\left(\frac{y}{x}\right) = \lambda^0 F_3(x, y)$$

$$F_4(\lambda x, \lambda y) = \sin \lambda x + \cos \lambda y \neq \lambda^n F_4(x, y), \text{ for any } n \in \mathbf{N}$$

Here, we observe that the functions F_1, F_2, F_3 can be written in the form $F(\lambda x, \lambda y) = \lambda^n F(x, y)$ but F_4 can not be written in this form. This leads to the following definition:

A function $F(x, y)$ is said to be *homogeneous function of degree n* if

$F(\lambda x, \lambda y) = \lambda^n F(x, y)$ for any nonzero constant λ.

We note that in the above examples, F_1, F_2, F_3 are homogeneous functions of degree 2, 1, 0 respectively but F_4 is not a homogeneous function.

We also observe that

$$F_1(x, y) = x^2\left(\frac{y^2}{x^2}+\frac{2y}{x}\right)=x^2h_1\left(\frac{y}{x}\right)$$

or $$F_1(x, y) = y^2\left(1+\frac{2x}{y}\right)=y^2h_2\left(\frac{x}{y}\right)$$

$$F_2(x, y) = x^1\left(2-\frac{3y}{x}\right)=x^1h_3\left(\frac{y}{x}\right)$$

or $$F_2(x, y) = y^1\left(2\frac{x}{y}-3\right)=y^1h_4\left(\frac{x}{y}\right)$$

$$F_3(x, y) = x^0\cos\left(\frac{y}{x}\right)=x^0\ h_5\left(\frac{y}{x}\right)$$

$$F_4(x, y) \neq x^n\ h_6\left(\frac{y}{x}\right), \text{ for any } n \in \mathbf{N}$$

or $$F_4\ (x, y) \neq y^n\ h_7\left(\frac{x}{y}\right), \text{ for any } n \in \mathbf{N}$$

Therefore, a function F (x, y) is a homogeneous function of degree n if

$$F(x, y) = x^n g\left(\frac{y}{x}\right) \quad \text{or} \quad y^n h\left(\frac{x}{y}\right)$$

A differential equation of the form $\frac{dy}{dx}=$ F (x, y) is said to be *homogenous* if F(x, y) is a homogenous function of degree zero.

To solve a homogeneous differential equation of the type

$$\frac{dy}{dx}=F(x,y) = g\left(\frac{y}{x}\right) \qquad \text{... (1)}$$

We make the substitution $y = v\,.\,x$ ... (2)

Differentiating equation (2) with respect to x, we get

$$\frac{dy}{dx} = v+x\frac{dv}{dx} \qquad \text{... (3)}$$

Substituting the value of $\frac{dy}{dx}$ from equation (3) in equation (1), we get

$$v + x\frac{dv}{dx} = g(v)$$

or $$x\frac{dv}{dx} = g(v) - v \qquad \text{... (4)}$$

Separating the variables in equation (4), we get

$$\frac{dv}{g(v) - v} = \frac{dx}{x} \qquad \text{... (5)}$$

Integrating both sides of equation (5), we get

$$\int \frac{dv}{g(v) - v} = \int \frac{1}{x}dx + \text{C} \qquad \text{... (6)}$$

Equation (6) gives general solution (primitive) of the differential equation (1) when we replace v by $\frac{y}{x}$.

Note If the homogeneous differential equation is in the form $\frac{dx}{dy} = \text{F}(x, y)$ where, F (x, y) is homogenous function of degree zero, then we make substitution $\frac{x}{y} = v$ i.e., $x = vy$ and we proceed further to find the general solution as discussed above by writing $\frac{dx}{dy} = \text{F}(x, y) = h\left(\frac{x}{y}\right)$.

Example 15 Show that the differential equation $(x - y)\frac{dy}{dx} = x + 2y$ is homogeneous and solve it.

Solution The given differential equation can be expressed as

$$\frac{dy}{dx} = \frac{x + 2y}{x - y} \qquad \text{... (1)}$$

Let $$\text{F}(x, y) = \frac{x \quad 2y}{x \quad y}$$

Now $$\text{F}(\lambda x, \lambda y) = \frac{(x \quad 2y)}{(x \quad y)} \quad {}^{0} F(x, y)$$

Therefore, F (x, y) is a homogenous function of degree zero. So, the given differential equation is a homogenous differential equation.

Alternatively,

$$\frac{dy}{dx} = \left(\frac{1+\frac{2y}{x}}{1-\frac{y}{x}}\right) = g\left(\frac{y}{x}\right) \qquad \text{... (2)}$$

R.H.S. of differential equation (2) is of the form $g \ \frac{y}{x}$ and so it is a homogeneous function of degree zero. Therefore, equation (1) is a homogeneous differential equation. To solve it we make the substitution

$$y = vx \qquad \text{... (3)}$$

Differentiating equation (3) with respect to, x we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \text{... (4)}$$

Substituting the value of y and $\frac{dy}{dx}$ in equation (1) we get

$$v + x\frac{dv}{dx} = \frac{1+2v}{1-v}$$

or
$$x\frac{dv}{dx} = \frac{1+2v}{1-v} - v$$

or
$$x\frac{dv}{dx} = \frac{v^2 \quad v \quad 1}{1 \quad v}$$

or
$$\frac{v \quad 1}{v^2 \quad v \quad 1}dv = \frac{dx}{x}$$

Integrating both sides of equation (5), we get

$$\frac{v \quad 1}{v^2 \quad v \quad 1}dv = \frac{dx}{x}$$

or
$$\frac{1}{2} \ \frac{2v \quad 1 \quad 3}{v^2 \quad v \quad 1}dv = -\log|x| + C_1$$

or $$\frac{1}{2}\quad\frac{2v\quad 1}{v^2\quad v\quad 1}dv\quad\frac{3}{2}\quad\frac{1}{v^2\quad v\quad 1}dv\qquad \log|x|\quad C_1$$

or $$\frac{1}{2}\log|v^2\quad v\quad 1|\quad\frac{3}{2}\quad\frac{1}{v\quad\frac{1}{2}^{\;2}\quad\frac{\sqrt{3}}{2}^{\;2}}dv\qquad \log|x|\quad C_1$$

or $$\frac{1}{2}\log|v^2\quad v\quad 1|\quad\frac{3}{2}.\frac{2}{\sqrt{3}}\tan^{\;1}\frac{2v\quad 1}{\sqrt{3}}\qquad \log|x|\quad C_1$$

or $$\frac{1}{2}\log|v^2\quad v\quad 1|\quad\frac{1}{2}\log x^2\quad\sqrt{3}\tan^{\;1}\frac{2v\quad 1}{\sqrt{3}}\quad C_1 \qquad \text{(Why?)}$$

Replacing v by $\frac{y}{x}$, we get

or $$\frac{1}{2}\log\left|\frac{y^2}{x^2}\quad\frac{y}{x}\quad 1\right|\quad\frac{1}{2}\log x^2\quad\sqrt{3}\tan^{\;1}\frac{2y\quad x}{\sqrt{3}x}\quad C_1$$

or $$\frac{1}{2}\log\left|\left(\frac{y^2}{x^2}+\frac{y}{x}+1\right)x^2\right|=\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)+C_1$$

or $$\log|(y^2+xy+x^2)|=2\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)+2C_1$$

or $$\log|(x^2+xy+y^2)|=2\sqrt{3}\tan^{-1}\left(\frac{x+2y}{\sqrt{3}x}\right)+C$$

which is the general solution of the differential equation (1)

Example 16 Show that the differential equation $x\cos\left(\frac{y}{x}\right)\frac{dy}{dx}=y\cos\left(\frac{y}{x}\right)+x$ is homogeneous and solve it.

Solution The given differential equation can be written as

$$\frac{dy}{dx}=\frac{y\cos\left(\frac{y}{x}\right)+x}{x\cos\left(\frac{y}{x}\right)} \qquad \text{... (1)}$$

It is a differential equation of the form $\frac{dy}{dx} = F(x, y)$.

Here
$$F(x, y) = \frac{y\cos\left(\frac{y}{x}\right) + x}{x\cos\left(\frac{y}{x}\right)}$$

Replacing x by λx and y by λy, we get

$$F(\lambda x, \lambda y) = \frac{\lambda\left[y\cos\left(\frac{y}{x}\right) + x\right]}{\lambda\left(x\cos\frac{y}{x}\right)} = \lambda^0 [F(x, y)]$$

Thus, $F(x, y)$ is a homogeneous function of degree zero.

Therefore, the given differential equation is a homogeneous differential equation. To solve it we make the substitution

$$y = vx \qquad \text{... (2)}$$

Differentiating equation (2) with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \text{... (3)}$$

Substituting the value of y and $\frac{dy}{dx}$ in equation (1), we get

$$v + x\frac{dv}{dx} = \frac{v\cos v + 1}{\cos v}$$

or
$$x\frac{dv}{dx} = \frac{v\cos v + 1}{\cos v} - v$$

or
$$x\frac{dv}{dx} = \frac{1}{\cos v}$$

or
$$\cos v\, dv = \frac{dx}{x}$$

Therefore
$$\int \cos v\, dv = \int \frac{1}{x}\, dx$$