Chapter 8

APPLICATION OF INTEGRALS

❖ *One should study Mathematics because it is only through Mathematics that nature can be conceived in harmonious form. – BIRKHOFF* ❖

8.1 Introduction

In geometry, we have learnt formulae to calculate areas of various geometrical figures including triangles, rectangles, trapezias and circles. Such formulae are fundamental in the applications of mathematics to many real life problems. The formulae of elementary geometry allow us to calculate areas of many simple figures. However, they are inadequate for calculating the areas enclosed by curves. For that we shall need some concepts of Integral Calculus.

A.L. Cauchy
(1789-1857)

In the previous chapter, we have studied to find the area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and x-axis, while calculating definite integral as the limit of a sum. Here, in this chapter, we shall study a specific application of integrals to find the area under simple curves, area between lines and arcs of circles, parabolas and ellipses (standard forms only). We shall also deal with finding the area bounded by the above said curves.

8.2 Area under Simple Curves

In the previous chapter, we have studied definite integral as the limit of a sum and how to evaluate definite integral using Fundamental Theorem of Calculus. Now, we consider the easy and intuitive way of finding the area bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$ and $x = b$. From Fig 8.1, we can think of area under the curve as composed of large number of very thin vertical strips. Consider an arbitrary strip of height y and width dx, then dA (area of the elementary strip) $= y dx$, where, $y = f(x)$.

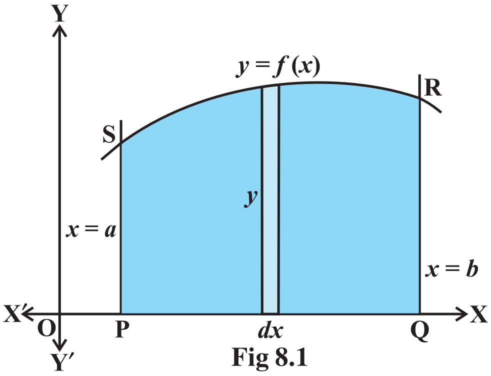

Fig 8.1

This area is called the *elementary area* which is located at an arbitrary position within the region which is specified by some value of x between a and b. We can think of the total area A of the region between x-axis, ordinates $x = a$, $x = b$ and the curve $y = f(x)$ as the result of adding up the elementary areas of thin strips across the region PQRSP. Symbolically, we express

$$A = \int_a^b dA = \int_a^b y\,dx = \int_a^b f(x)\,dx$$

The area A of the region bounded by the curve $x = g(y)$, y-axis and the lines $y = c$, $y = d$ is given by

$$A = \int_c^d x\,dy = \int_c^d g(y)\,dy$$

Here, we consider horizontal strips as shown in the Fig 8.2

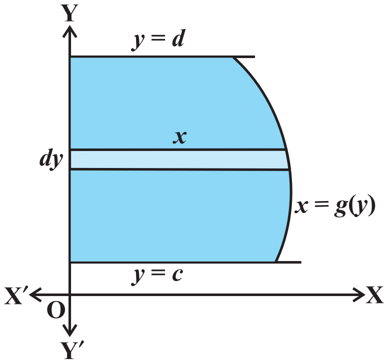

Fig 8.2

Remark If the position of the curve under consideration is below the x-axis, then since $f(x) < 0$ from $x = a$ to $x = b$, as shown in Fig 8.3, the area bounded by the curve, x-axis and the ordinates $x = a$, $x = b$ come out to be negative. But, it is only the numerical value of the area which is taken into consideration. Thus, if the area is negative, we take its absolute value, i.e., $\left|\int_a^b f(x)\,dx\right|$.

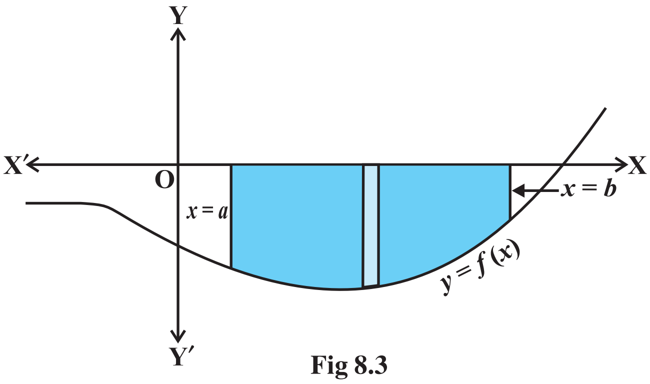

Fig 8.3

Generally, it may happen that some portion of the curve is above x-axis and some is below the x-axis as shown in the Fig 8.4. Here, $A_1 < 0$ and $A_2 > 0$. Therefore, the area A bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$ and $x = b$ is given by $A = |A_1| + A_2$.

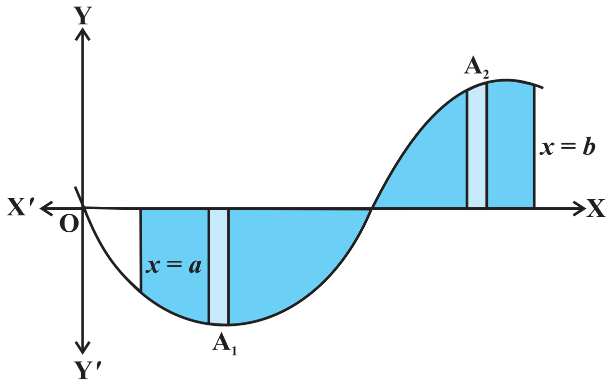

Fig 8.4

Example 1 Find the area enclosed by the circle $x^2 + y^2 = a^2$.

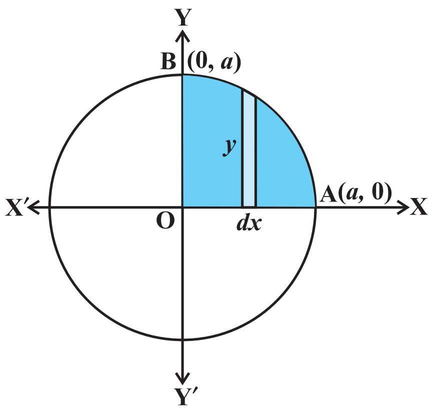

Fig 8.5

Solution From Fig 8.5, the whole area enclosed by the given circle

= 4 (area of the region AOBA bounded by the curve, x-axis and the ordinates $x = 0$ and $x = a$) [as the circle is symmetrical about both x-axis and y-axis]

$$= 4\int_0^a y\,dx \text{ (taking vertical strips)}$$

$$= 4\int_0^a \sqrt{a^2 - x^2}\,dx$$

Since $x^2 + y^2 = a^2$ gives $y = \pm\sqrt{a^2 - x^2}$

As the region AOBA lies in the first quadrant, y is taken as positive. Integrating, we get the whole area enclosed by the given circle

$$= 4\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= 4\left[\left(\frac{a}{2}\times 0 + \frac{a^2}{2}\sin^{-1}1\right) - 0\right] = 4\left(\frac{a^2}{2}\right)\left(\frac{\pi}{2}\right) = \pi a^2$$

Alternatively, considering horizontal strips as shown in Fig 8.6, the whole area of the region enclosed by circle

$$= 4\int_0^a x dy = 4\int_0^a \sqrt{a^2 - y^2}\, dy \qquad \text{(Why?)}$$

$$= 4\left[\frac{y}{2}\sqrt{a^2 - y^2} + \frac{a^2}{2}\sin^{-1}\frac{y}{a}\right]_0^a$$

$$= 4\left[\left(\frac{a}{2}\times 0 + \frac{a^2}{2}\sin^{-1}1\right) - 0\right]$$

$$= 4\frac{a^2}{2}\frac{\pi}{2} = \pi a^2$$

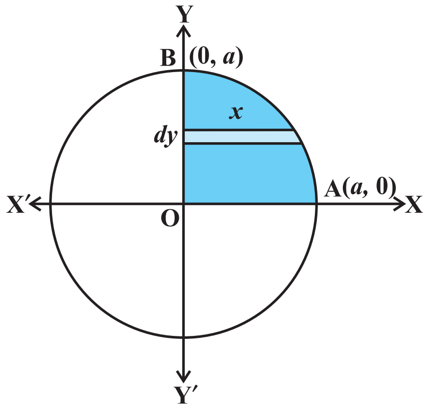

Fig 8.6

Example 2 Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution From Fig 8.7, the area of the region ABA′B′A bounded by the ellipse

$$= 4\left(\begin{array}{l}\textit{area of the region AOBA in the first quadrant bounded}\\ \textit{by the curve, } x-\textit{axis and the ordinates } x = 0, x = a\end{array}\right)$$

(as the ellipse is symmetrical about both *x*-axis and *y*-axis)

$$= 4\int_0^a y dx \quad \text{(taking vertical strips)}$$

Now $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ gives $y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$, but as the region AOBA lies in the first quadrant, *y* is taken as positive. So, the required area is

$$= 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a \quad \text{(Why?)}$$

$$= \frac{4b}{a}\left[\left(\frac{a}{2}\times 0 + \frac{a^2}{2}\sin^{-1}1\right) - 0\right]$$

$$= \frac{4b}{a}\frac{a^2}{2}\frac{\pi}{2} = \pi ab$$

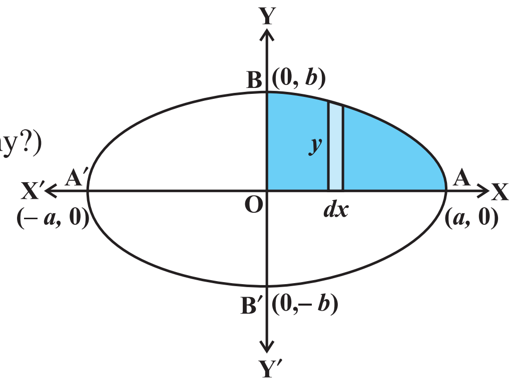

Fig 8.7

Alternatively, considering horizontal strips as shown in the Fig 8.8, the area of the ellipse is

$$= 4\int_0^b x\,dy = 4\frac{a}{b}\int_0^b \sqrt{b^2-y^2}\,dy \quad \text{(Why?)}$$

$$= \frac{4a}{b}\left[\frac{y}{2}\sqrt{b^2-y^2}+\frac{b^2}{2}\sin^{-1}\frac{y}{b}\right]_0^b$$

$$= \frac{4a}{b}\left[\left(\frac{b}{2}\times 0+\frac{b^2}{2}\sin^{-1}1\right)-0\right]$$

$$= \frac{4a}{b}\frac{b^2}{2}\frac{\pi}{2} = \pi ab$$

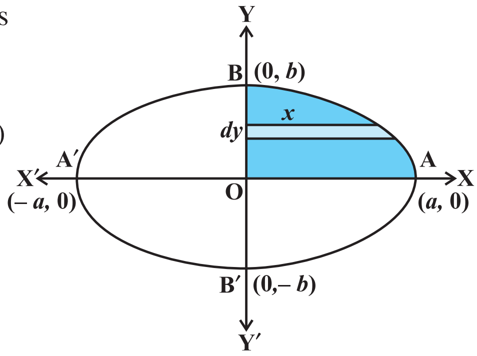

Fig 8.8

8.2.1 *The area of the region bounded by a curve and a line*

In this subsection, we will find the area of the region bounded by a line and a circle, a line and a parabola, a line and an ellipse. Equations of above mentioned curves will be in their standard forms only as the cases in other forms go beyond the scope of this textbook.

Example 3 Find the area of the region bounded by the curve $y = x^2$ and the line $y = 4$.

Solution Since the given curve represented by the equation $y = x^2$ is a parabola symmetrical about y-axis only, therefore, from Fig 8.9, the required area of the region AOBA is given by

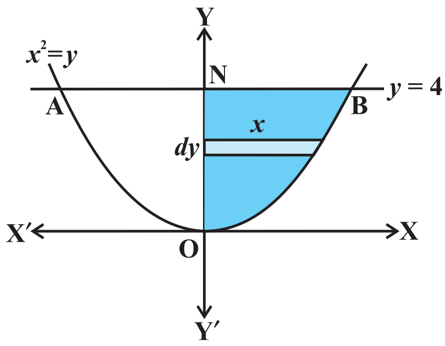

Fig 8.9

$$2\int_0^4 x\,dy =$$

$$2\begin{pmatrix}\text{area of the region BONB bounded by curve, } y-\text{axis}\\ \text{and the lines } y=0 \text{ and } y=4\end{pmatrix}$$

$$= 2\int_0^4 \sqrt{y}\,dy = 2\times\frac{2}{3}\left[y^{\frac{3}{2}}\right]_0^4 = \frac{4}{3}\times 8 = \frac{32}{3} \quad \text{(Why?)}$$

Here, we have taken horizontal strips as indicated in the Fig 8.9.

Alternatively, we may consider the vertical strips like PQ as shown in the Fig 8.10 to obtain the area of the region AOBA. To this end, we solve the equations $x^2 = y$ and $y = 4$ which gives $x = -2$ and $x = 2$.

Thus, the region AOBA may be stated as the region bounded by the curve $y = x^2$, $y = 4$ and the ordinates $x = -2$ and $x = 2$.

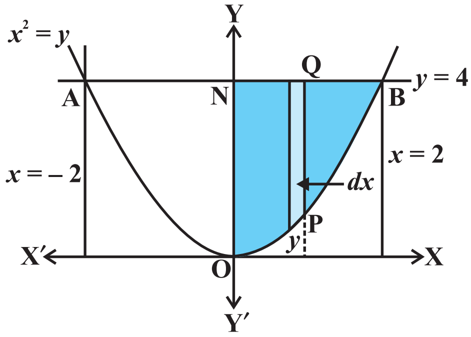

Fig 8.10

Therefore, the area of the region AOBA

$$= \int_{-2}^{2} y\,dx$$

[y = (y-coordinate of Q) – (y-coordinate of P) = $4 - x^2$]

$$= 2\int_{0}^{2} \left(4 - x^2\right) dx \qquad \text{(Why?)}$$

$$= 2\left[4x - \frac{x^3}{3}\right]_0^2 = 2\left[4 \times 2 - \frac{8}{3}\right] = \frac{32}{3}$$

Remark From the above examples, it is inferred that we can consider either vertical strips or horizontal strips for calculating the area of the region. Henceforth, we shall consider either of these two, most preferably vertical strips.

Example 4 Find the area of the region in the first quadrant enclosed by the x-axis, the line $y = x$, and the circle $x^2 + y^2 = 32$.

Solution The given equations are

$$y = x \qquad \text{... (1)}$$

and
$$x^2 + y^2 = 32 \qquad \text{... (2)}$$

Solving (1) and (2), we find that the line and the circle meet at B(4, 4) in the first quadrant (Fig 8.11). Draw perpendicular BM to the x-axis.

Therefore, the required area = area of the region OBMO + area of the region BMAB.

Now, the area of the region OBMO

$$= \int_0^4 y\,dx = \int_0^4 x\,dx \qquad \text{... (3)}$$

$$= \frac{1}{2}\left[x^2\right]_0^4 = 8$$

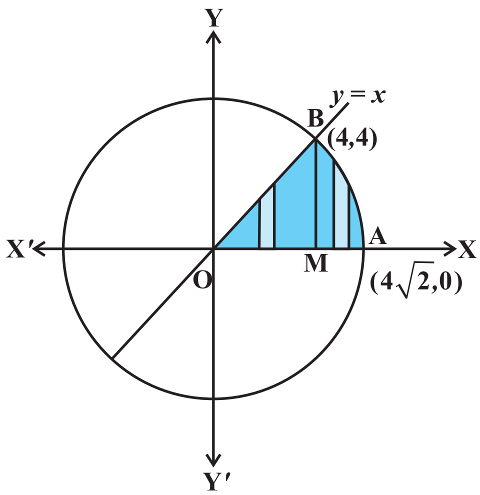

Fig 8.11

Again, the area of the region BMAB

$$= \int_4^{4\sqrt{2}} y\,dx = \int_4^{4\sqrt{2}} \sqrt{32 - x^2}\,dx$$

$$= \left[\frac{1}{2} x\sqrt{32 - x^2} + \frac{1}{2} \times 32 \times \sin^{-1} \frac{x}{4\sqrt{2}}\right]_4^{4\sqrt{2}}$$

$$= \left(\frac{1}{2} 4\sqrt{2} \times 0 + \frac{1}{2} \times 32 \times \sin^{-1} 1\right) - \left(\frac{4}{2}\sqrt{32 - 16} + \frac{1}{2} \times 32 \times \sin^{-1} \frac{1}{\sqrt{2}}\right)$$

$$= 8\pi - (8 + 4\pi) = 4\pi - 8 \qquad \text{... (4)}$$

Adding (3) and (4), we get, the required area = 4π.

Example 5 Find the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the ordinates $x = 0$ and $x = ae$, where, $b^2 = a^2 (1 - e^2)$ and $e < 1$.

Solution The required area (Fig 8.12) of the region BOB′RFSB is enclosed by the ellipse and the lines $x = 0$ and $x = ae$.

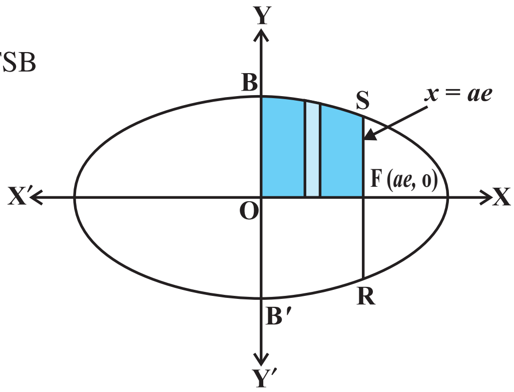

Fig 8.12

Note that the area of the region BOB′RFSB

$$= 2\int_0^{ae} y\,dx = 2\frac{b}{a}\int_0^{ae} \sqrt{a^2 - x^2}\,dx$$

$$= \frac{2b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^{ae}$$

$$= \frac{2b}{2a}\left[ae\sqrt{a^2 - a^2e^2} + a^2\sin^{-1} e\right]$$

$$= ab\left[e\sqrt{1 - e^2} + \sin^{-1} e\right]$$

EXERCISE 8.1

1. Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1$, $x = 4$ and the x-axis in the first quadrant.
2. Find the area of the region bounded by $y^2 = 9x$, $x = 2$, $x = 4$ and the x-axis in the first quadrant.

3. Find the area of the region bounded by $x^2 = 4y$, $y = 2$, $y = 4$ and the y-axis in the first quadrant.

4. Find the area of the region bounded by the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$.

5. Find the area of the region bounded by the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$.

6. Find the area of the region in the first quadrant enclosed by x-axis, line $x = \sqrt{3}\,y$ and the circle $x^2 + y^2 = 4$.

7. Find the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line $x=\frac{a}{\sqrt{2}}$.

8. The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a.

9. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$.

10. Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

11. Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.

Choose the correct answer in the following Exercises 12 and 13.

12. Area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $x = 2$ is

    (A) π (B) $\frac{\pi}{2}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{4}$

13. Area of the region bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$ is

    (A) 2 (B) $\frac{9}{4}$ (C) $\frac{9}{3}$ (D) $\frac{9}{2}$

8.3 Area between Two Curves

Intuitively, true in the sense of Leibnitz, integration is the act of calculating the area by cutting the region into a large number of small strips of elementary area and then adding up these elementary areas. Suppose we are given two curves represented by $y = f(x)$, $y = g(x)$, where $f(x) \geq g(x)$ in $[a, b]$ as shown in Fig 8.13. Here the points of intersection of these two curves are given by $x = a$ and $x = b$ obtained by taking common values of y from the given equation of two curves.

For setting up a formula for the integral, it is convenient to take elementary area in the form of vertical strips. As indicated in the Fig 8.13, elementary strip has height

$f(x) - g(x)$ and width dx so that the elementary area

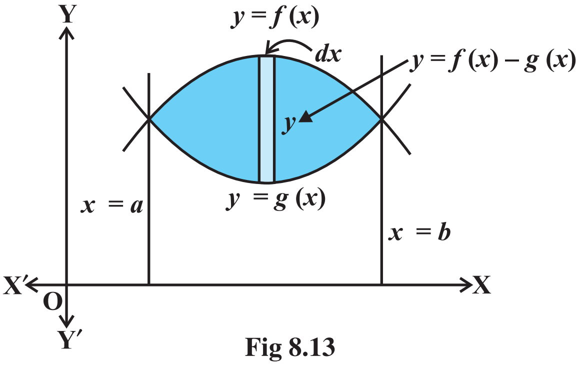

Fig 8.13

$dA = [f(x) - g(x)]\, dx$, and the total area A can be taken as

$$A = \int_a^b [f(x) - g(x)]\, dx$$

Alternatively,

A = [area bounded by $y = f(x)$, x-axis and the lines $x = a$, $x = b$]

– [area bounded by $y = g(x)$, x-axis and the lines $x = a$, $x = b$]

$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b [f(x) - g(x)]\,dx, \text{ where } f(x) \geq g(x) \text{ in } [a, b]$$

If $f(x) \geq g(x)$ in $[a, c]$ and $f(x) \leq g(x)$ in $[c, b]$, where $a < c < b$ as shown in the Fig 8.14, then the area of the regions bounded by curves can be written as

Total Area = Area of the region ACBDA + Area of the region BPRQB

$$= \int_a^c [f(x) - g(x)]\,dx + \int_c^b [g(x) - f(x)]\,dx$$

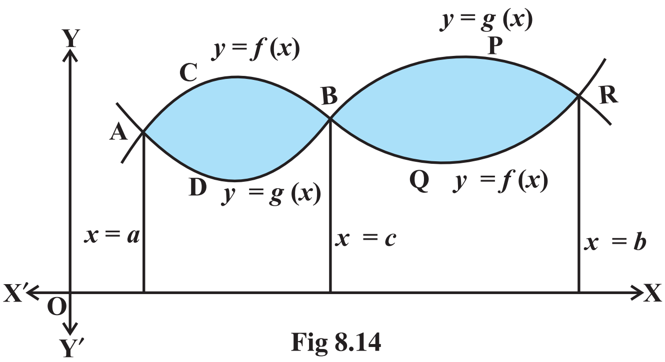

Fig 8.14

Example 6 Find the area of the region bounded by the two parabolas $y = x^2$ and $y^2 = x$.

Solution The point of intersection of these two parabolas are O (0, 0) and A (1, 1) as shown in the Fig 8.15.

Here, we can set $y^2 = x$ or $y = \sqrt{x} = f(x)$ and $y = x^2 = g(x)$, where, $f(x) \geq g(x)$ in [0, 1].

Therefore, the required area of the shaded region

$$= \int_0^1 [f(x) - g(x)]\, dx$$

$$= \int_0^1 \left[\sqrt{x} - x^2\right] dx = \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

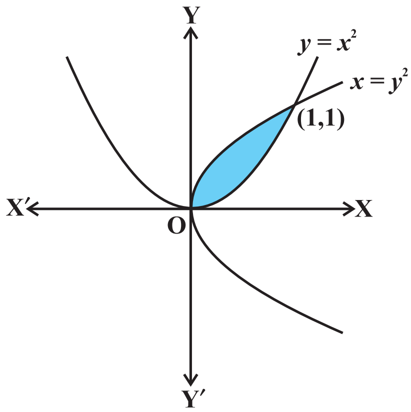

Fig 8.15

Example 7 Find the area lying above x-axis and included between the circle $x^2 + y^2 = 8x$ and inside of the parabola $y^2 = 4x$.

Solution The given equation of the circle $x^2 + y^2 = 8x$ can be expressed as $(x - 4)^2 + y^2 = 16$. Thus, the centre of the circle is (4, 0) and radius is 4. Its intersection with the parabola $y^2 = 4x$ gives

$$x^2 + 4x = 8x$$

or $\quad x^2 - 4x = 0$

or $\quad x(x - 4) = 0$

or $\quad x = 0, x = 4$

Thus, the points of intersection of these two curves are O(0,0) and P(4,4) above the x-axis.

From the Fig 8.16, the required area of the region OPQCO included between these two curves above x-axis is

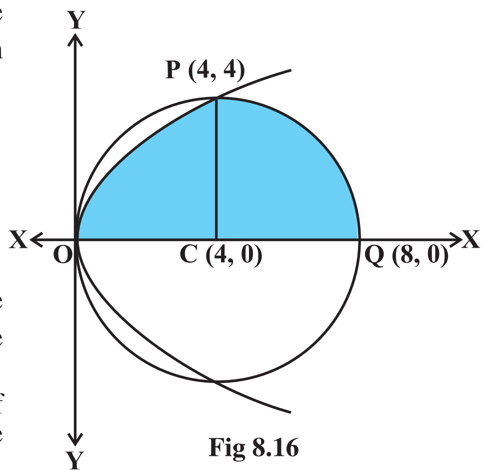

Fig 8.16

= (area of the region OCPO) + (area of the region PCQP)

$$= \int_0^4 y\,dx + \int_4^8 y\,dx$$

$$= 2\int_0^4 \sqrt{x}\,dx + \int_4^8 \sqrt{4^2 - (x-4)^2}\,dx \quad \text{(Why?)}$$

$$= 2\times\frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^4 + \int_0^4 \sqrt{4^2-t^2}\,dt,\ \text{where, } x-4=t \qquad \text{(Why?)}$$

$$= \frac{32}{3} + \left[\frac{t}{2}\sqrt{4^2-t^2} + \frac{1}{2}\times 4^2 \times \sin^{-1}\frac{t}{4}\right]_0^4$$

$$= \frac{32}{3} + \left[\frac{4}{2}\times 0 + \frac{1}{2}\times 4^2 \times \sin^{-1}1\right] = \frac{32}{3} + \left[0 + 8\times\frac{\pi}{2}\right] = \frac{32}{3} + 4\pi = \frac{4}{3}(8+3\pi)$$

Example 8 In Fig 8.17, AOBA is the part of the ellipse $9x^2 + y^2 = 36$ in the first quadrant such that OA = 2 and OB = 6. Find the area between the arc AB and the chord AB.

Solution Given equation of the ellipse $9x^2 + y^2 = 36$ can be expressed as $\frac{x^2}{4} + \frac{y^2}{36} = 1$ or $\frac{x^2}{2^2} + \frac{y^2}{6^2} = 1$ and hence, its shape is as given in Fig 8.17.

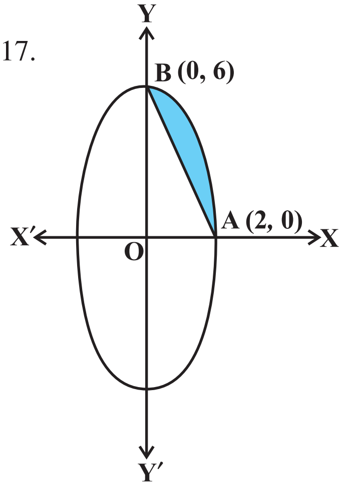

Fig 8.17

Accordingly, the equation of the chord AB is

$$y - 0 = \frac{6-0}{0-2}(x-2)$$

or $\qquad y = -3(x-2)$

or $\qquad y = -3x + 6$

Area of the shaded region as shown in the Fig 8.17.

$$= 3\int_0^2 \sqrt{4-x^2}\,dx - \int_0^2 (6-3x)\,dx \qquad \text{(Why?)}$$

$$= 3\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 - \left[6x - \frac{3x^2}{2}\right]_0^2$$

$$= 3\left[\frac{2}{2}\times 0 + 2\sin^{-1}(1)\right] - \left[12 - \frac{12}{2}\right] = 3\times 2\times\frac{\pi}{2} - 6 = 3\pi - 6$$

Example 9 Using integration find the area of region bounded by the triangle whose vertices are (1, 0), (2, 2) and (3, 1).

Solution Let A(1, 0), B(2, 2) and C(3, 1) be the vertices of a triangle ABC (Fig 8.18).

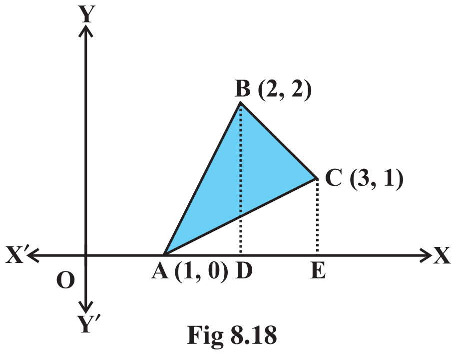

Fig 8.18

Area of ΔABC

= Area of ΔABD + Area of trapezium BDEC – Area of ΔAEC

Now equation of the sides AB, BC and CA are given by

$$y = 2(x-1),\ y = 4 - x,\ y = \frac{1}{2}(x-1), \text{ respectively.}$$

Hence, area of Δ ABC $= \int_1^2 2(x-1)\,dx + \int_2^3 (4-x)\,dx - \int_1^3 \frac{x-1}{2}\,dx$

$$= 2\left[\frac{x^2}{2} - x\right]_1^2 + \left[4x - \frac{x^2}{2}\right]_2^3 - \frac{1}{2}\left[\frac{x^2}{2} - x\right]_1^3$$

$$= 2\left[\left(\frac{2^2}{2} - 2\right) - \left(\frac{1}{2} - 1\right)\right] + \left[\left(4\times3 - \frac{3^2}{2}\right) - \left(4\times2 - \frac{2^2}{2}\right)\right] - \frac{1}{2}\left[\left(\frac{3^2}{2} - 3\right) - \left(\frac{1}{2} - 1\right)\right]$$

$$= \frac{3}{2}$$

Example 10 Find the area of the region enclosed between the two circles: $x^2 + y^2 = 4$ and $(x-2)^2 + y^2 = 4$.

Solution Equations of the given circles are

$$x^2 + y^2 = 4 \quad \ldots (1)$$

and $$(x-2)^2 + y^2 = 4 \quad \ldots (2)$$

Equation (1) is a circle with centre O at the origin and radius 2. Equation (2) is a circle with centre C (2, 0) and radius 2. Solving equations (1) and (2), we have

$$(x-2)^2 + y^2 = x^2 + y^2$$

or $$x^2 - 4x + 4 + y^2 = x^2 + y^2$$

or $x = 1$ which gives $y = \pm\sqrt{3}$

Thus, the points of intersection of the given circles are A$(1, \sqrt{3})$ and A′$(1, -\sqrt{3})$ as shown in the Fig 8.19.

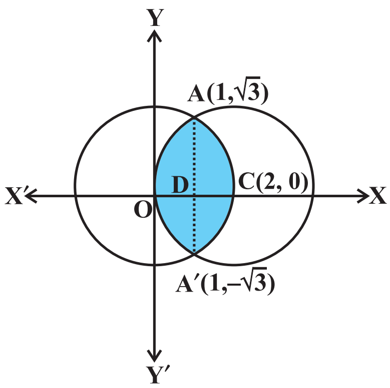

Fig 8.19

Required area of the enclosed region OACA′O between circles

$= 2$ [area of the region ODCAO] (Why?)

$= 2$ [area of the region ODAO + area of the region DCAD]

$$= 2\left[\int_0^1 y\,dx + \int_1^2 y\,dx\right]$$

$$= 2\left[\int_0^1 \sqrt{4-(x-2)^2}\,dx + \int_1^2 \sqrt{4-x^2}\,dx\right] \quad \text{(Why?)}$$

$$= 2\left[\frac{1}{2}(x-2)\sqrt{4-(x-2)^2} + \frac{1}{2}\times 4\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1$$

$$+\ 2\left[\frac{1}{2}x\sqrt{4-x^2} + \frac{1}{2}\times 4\sin^{-1}\frac{x}{2}\right]_1^2$$

$$= \left[(x-2)\sqrt{4-(x-2)^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1 + \left[x\sqrt{4-x^2} + 4\sin^{-1}\frac{x}{2}\right]_1^2$$

$$= \left[\left(-\sqrt{3} + 4\sin^{-1}\left(\frac{-1}{2}\right)\right) - 4\sin^{-1}(-1)\right] + \left[4\sin^{-1}1 - \sqrt{3} - 4\sin^{-1}\frac{1}{2}\right]$$

$$= \left[\left(-\sqrt{3} - 4\times\frac{\pi}{6}\right) + 4\times\frac{\pi}{2}\right] + \left[4\times\frac{\pi}{2} - \sqrt{3} - 4\times\frac{\pi}{6}\right]$$

$$= \left(-\sqrt{3} - \frac{2\pi}{3} + 2\pi\right) + \left(2\pi - \sqrt{3} - \frac{2\pi}{3}\right)$$

$$= \frac{8\pi}{3} - 2\sqrt{3}$$

EXERCISE 8.2

1. Find the area of the circle $4x^2 + 4y^2 = 9$ which is interior to the parabola $x^2 = 4y$.
2. Find the area bounded by curves $(x - 1)^2 + y^2 = 1$ and $x^2 + y^2 = 1$.
3. Find the area of the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 3$.
4. Using integration find the area of region bounded by the triangle whose vertices are $(-1, 0)$, $(1, 3)$ and $(3, 2)$.
5. Using integration find the area of the triangular region whose sides have the equations $y = 2x + 1$, $y = 3x + 1$ and $x = 4$.

Choose the correct answer in the following exercises 6 and 7.

6. Smaller area enclosed by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$ is

(A) $2(\pi - 2)$ (B) $\pi - 2$ (C) $2\pi - 1$ (D) $2(\pi + 2)$

7. Area lying between the curves $y^2 = 4x$ and $y = 2x$ is

(A) $\frac{2}{3}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) $\frac{3}{4}$

Miscellaneous Examples

Example 11 Find the area of the parabola $y^2 = 4ax$ bounded by its latus rectum.

Solution From Fig 8.20, the vertex of the parabola $y^2 = 4ax$ is at origin (0, 0). The equation of the latus rectum LSL′ is $x = a$. Also, parabola is symmetrical about the x-axis.

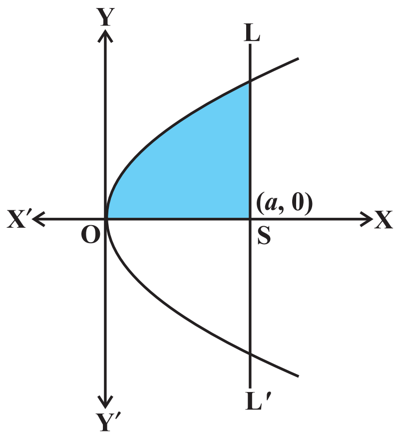

Fig 8.20

The required area of the region OLL′O

= 2 (area of the region OLSO)

$$= 2\int_0^a y\,dx = 2\int_0^a \sqrt{4ax}\,dx$$

$$= 2 \times 2\sqrt{a}\int_0^a \sqrt{x}\,dx$$

$$= 4\sqrt{a} \times \frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^a$$

$$= \frac{8}{3}\sqrt{a}\left[a^{\frac{3}{2}}\right] = \frac{8}{3}a^2$$

Example 12 Find the area of the region bounded by the line $y = 3x + 2$, the x-axis and the ordinates $x = -1$ and $x = 1$.

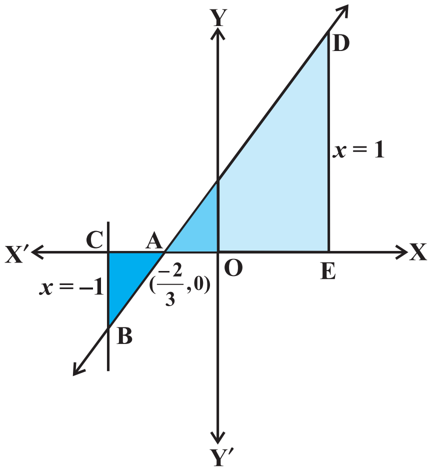

Fig 8.21

Solution As shown in the Fig 8.21, the line $y = 3x + 2$ meets x-axis at $x = \frac{-2}{3}$ and its graph lies below x-axis for $x \in \left(-1, \frac{-2}{3}\right)$ and above x-axis for $x \in \left(\frac{-2}{3}, 1\right)$.